SOCIETY OF ACTUARIES®

**Advanced Long-Term Actuarial
Mathematics Exam**

Exam ALTAM

Date: Tuesday, April 25, 2023

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 6 questions numbered 1 through 6 with a total of 60 points.

   The points for each question are indicated at the beginning of the question.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond these Instructions to Candidates.

3. Each question part or subpart should be answered in the provided Yellow Answer Booklet. Graders will only look at the work in the Yellow Answer Booklet.

4. The Excel file that contains the tables will **not** be uploaded for grading, and therefore will NOT BE GRADED. It should be used for looking up values for statistical functions and may be used for calculations.

5. If you use Excel for calculations, you should include as much information in the Yellow Answer Booklet as if you had used a calculator, including formulas and intermediate calculations where relevant. Written answers without sufficient support will not receive full credit.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas in the Yellow Answer Booklet. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, hand in all your written answer sheets to the Prometric center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

1.

(*10 points*) You are given the following information about a portfolio of in-force policies.

(i)   Policyholders are either Healthy (State 0), Temporarily Disabled (State 1), Permanently Disabled (State 2), or Dead (State 3).

(ii)   At time 0, there are 14 Healthy lives, 0 Temporarily Disabled lives, and 1 Permanently Disabled life.

(iii)   All 15 lives are independent.

(iv)   Transitions are modeled as a homogeneous Markov model.

(v)   The annual transition matrix is

$$P = \begin{pmatrix} 0.80 & 0.10 & 0.05 & 0.05 \\ 0.20 & 0.50 & 0.10 & 0.20 \\ 0 & 0 & 0.70 & 0.30 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(vi)   The two-year transition matrix, but with some missing values denoted by $--$, is

$$_2P = \begin{pmatrix} 0.660 & -- & 0.085 & -- \\ 0.260 & 0.270 & 0.130 & -- \\ 0 & 0 & -- & -- \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a)   (*2 points*) Calculate the missing values in the two-year transition matrix.

(b)   (*2 points*) Calculate the probability that exactly one policyholder will die within 2 years.

(c)   (*1 point*) Calculate the expected number of policyholders who will die within 2 years.

1.   Continued

(d)   (*4 points*) The policies provide the following benefits:
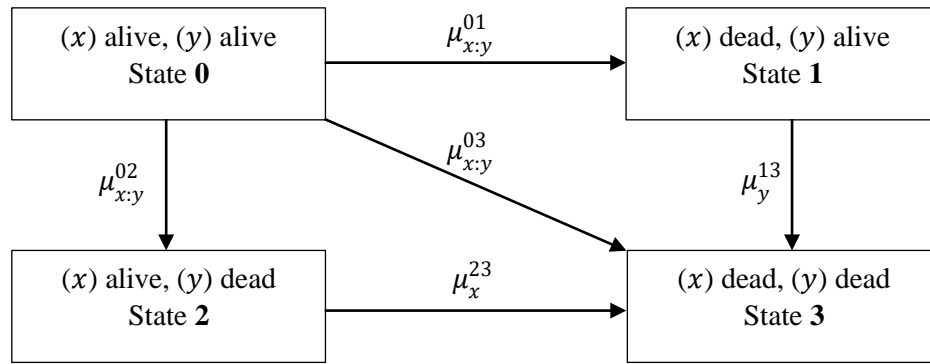
- A benefit of 500,000 is paid at the end of the year that a policyholder becomes permanently disabled, if they are still alive at that time.

- A benefit of 1,000,000 is paid at the end of the year of death (whether they were disabled or not).

The insurer values the benefits at $i = 0.06$.

(i)   (*1.5 points*) Show that the expected present value of the benefits payable in the next 2 years to the one Permanently Disabled life is 469,920 to the nearest 10. You should calculate the value to the nearest 1.

(ii)   (*2.5 points*) Calculate the expected present value of all the benefits payable in the next 2 years for the 14 Healthy lives.

(e)   (*1 point*) Describe one disadvantage of the assumption of homogeneity of the Markov model for valuing this portfolio.

2.

(*11 points*) Consider the following multi-state model for modeling joint life insurance policies issued to a couple (x) and (y):

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┌─────────────────────┐      μ^{01}_{x:y}      ┌─────────────────────┐
│  (x) alive, (y) alive│ ───────────────────▶ │  (x) dead, (y) alive │
│      State 0         │                       │      State 1         │
└─────────────────────┘                       └─────────────────────┘
        │           ╲ μ^{03}_{x:y}                      │
 μ^{02}_{x:y}        ╲                            μ^{13}_y
        ▼             ╲                                 ▼
┌─────────────────────┐  μ^{23}_x  ┌─────────────────────┐
│  (x) alive, (y) dead │ ─────────▶ │  (x) dead, (y) dead  │
│      State 2         │            │      State 3         │
└─────────────────────┘            └─────────────────────┘
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(a)   (*1 point*) State two reasons why a couple may have dependent future lifetimes.

(b)   (*1 point*) State the conditions for (x) and (y) to have independent future lifetimes, in terms of $\mu^{01}_{x+t:y+t}$, $\mu^{02}_{x+t:y+t}$, μ^{13}_{y+t}, μ^{23}_{x+t}, and $\mu^{03}_{x+t:y+t}$.

(c)   (*3 points*)

    (i)   Write down the Kolmogorov forward differential equation, with the associated boundary condition, for $_tp^{00}_{x:y}$.

    (ii)   Use the Kolmogorov forward differential equation in part (i) to show that

$$_tp^{00}_{x:y} = \exp\left(-\int_0^t \left(\mu^{01}_{x+u:y+u} + \mu^{02}_{x+u:y+u} + \mu^{03}_{x+u:y+u} \right) du \right)$$

2. **Continued**

A couple, (x) who is age 40 and (y) who is age 50, buys a fully discrete last survivor whole life insurance with a sum insured of 100,000.

You are given:

      (i)   The couple's future lifetimes are independently distributed.

      (ii)   Premiums are payable while at least one life is alive for a maximum of 10 years.

      (iii)   Mortality follows the Standard Ultimate Life Table (SULT).

      (iv) $i = 0.05$

(d)   (*2 points*) Show that the annual net premium is 1,220 to the nearest 10. You should calculate the value to the nearest 1.

(e)   (*2 points*)

      (i)   Show that the policy value at time 10, if (y) is alive but (x) is dead, is 29,000 to the nearest 100. You should calculate the value to the nearest 1.

      (ii)   Show that the policy value at time 10, if both (x) and (y) are alive, is 15,900 to the nearest 100. You should calculate the value to the nearest 1.

(f)   (*2 points*) The insurer decides to incorporate a common shock risk in the joint life model, although each life's marginal individual mortality still follows the SULT.

      (i)   State with reasons whether the policy value in (e)(i) would increase, decrease, or stay the same as a result of the common shock risk.

      (ii)   State with reasons whether the policy value in (e)(ii) would increase, decrease, or stay the same as a result of the common shock risk.

3.

(*10 points*) You are conducting a profit test of a Type A universal life insurance policy with a face amount of 150,000 issued to (40).

You are given the following information used to project the policy account values, assuming the policy remains in force.

Policy year k	Annual premium	Percent of premium charge	Annual expense charge	COI rate per 1000 of insurance	Annual interest rate for discounting the COI	Annual credited interest rate	Corridor factor
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	7,000	6%	55	2.0	5%	6%	1.97
10	7,000	6%	55	2.2	5%	6%	1.91

You are also given that the projected account value at the end of year 8 is 62,000.

(a) (*4 points*)

(i) Show that the projected account value at the end of year 9 is 72,480 to the nearest 10. You should calculate the value to the nearest 1.

(ii) Show that the projected account value at the end of year 10 is 83,580 to the nearest 10. You should calculate the value to the nearest 1.

You are given the following profit test assumptions.

(i) The interest earned on all insurer's funds is 7.5% per year.

(ii) At the end of each of the first 9 policy years, 2% of the in-force policyholders surrender.

(iii) At the end of policy year 10, 15% of the in-force policyholders surrender.

(iv) The surrender charge for policy year 10 is 1,500.

(v) Mortality experience follows the Standard Ultimate Life Table.

(vi) Commission expenses are 5% of premiums.

(vii) Other incurred expenses are 45 at the start of each year, 100 on surrender, and 250 at the end of the year of death.

(viii) There are no reserves held other than the account value.

3.   Continued

(b)   (*1 point*) Show that the expected cost of death benefit plus associated expenses payable at the end of year 10, given that the policy is in force at the start of the year, is 176 to the nearest 1. You should calculate the value to the nearest 0.1.

(c)   (*3 points*) Show that the expected surplus emerging at the end of year 10, given that the policy is in force at the start of the year, is 1,570 to the nearest 10. You should calculate the value to the nearest 1.

(d)   (*1 point*) Calculate the expected surplus emerging at the end of year 10, per policy issued, Π_{10}.

(e)   (*1 point*) Describe two reasons why the insurer might hold reserves higher than the account value.

4.

(*10 points*) A Continuing Care Retirement Community (CCRC) offers new residents Modified Life Care contracts.

The CCRC contains three different types of accommodation: Independent Living Units (ILU), Assisted Living Units (ALU), and Specialized Nursing Facilities (SNF). The following Markov model is used to determine fees.
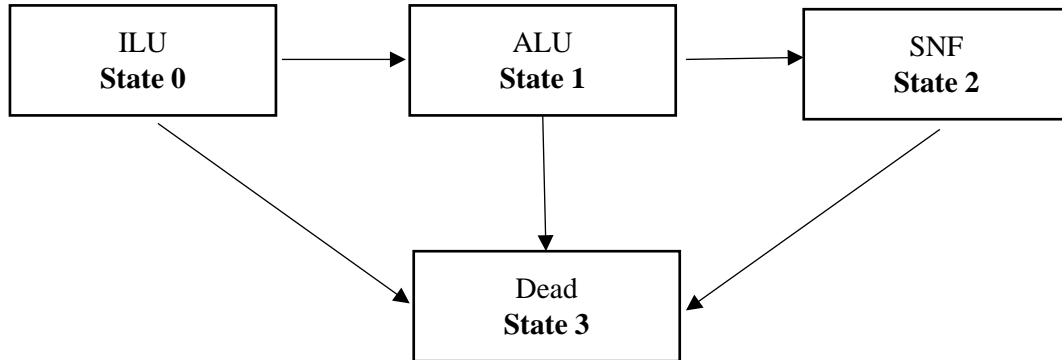
You are given:

(i)   The CCRC incurs the following monthly costs per resident, payable continuously, by type of accommodation.

   ILU: 3,000
   ALU: 7,000
   SNF: 15,000

(ii)   Each resident pays a lump sum fee immediately on entry and pays residence fees continuously while in the CCRC.

(iii)   Residence fees are a fixed proportion, γ, of the monthly costs.

(iv)   The value of γ is determined such that the EPV of future costs is equal to the EPV of all fees, including the entry fee.

(v) $i = 0.05$

(vi)   The following actuarial functions, evaluated at 5%.

x	$\bar{a}_x^{00}$	$\bar{a}_x^{01}$	$\bar{a}_x^{02}$	$\bar{a}_x^{11}$	$\bar{a}_x^{12}$	$\bar{a}_x^{22}$
65	11.4106	1.3570	0.3745	11.8352	0.7979	10.6905
70	9.5210	1.7037	0.4942	10.1754	0.9960	9.1961

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4. Continued

An individual enters the CCRC, in an ILU, at age 65. Their entry fee is 150,000.

(a)   (*1 point*) Show that the expected present value of future costs is 592,200 to the nearest 100. You should calculate the value to the nearest 10.

(b)   (*2 points*) Show that $\gamma = 0.75$ to the nearest 0.01. You should calculate the value to the nearest 0.001.

Let $_{t}V_{x}^{(j)}$ denote the EPV at time t of the CCRC's future costs minus future income, for an individual who entered the CCRC at age x, and who is now in state j at age $x+t$.

(c)   (*2 points*) Show that $_{5}V_{65}^{(1)}$ is 261,900 to the nearest 100. You should calculate the value to the nearest 10.

You are also given:

- $_{5}V_{65}^{(2)} = 419,280$
- $\mu_{70}^{12} = 0.00156$ and $\mu_{70}^{13} = 0.01194$

(d)   (*4 points*)

   (i)   (*1 point*) Write down Thiele's differential equation for $_{t}V_{65}^{(1)}$.

   (ii)   (*1 point*) Show that $\dfrac{d}{dt}\left(_{t}V_{65}^{(1)} \right)$ evaluated at $t = 5$ is $-5,620$ to the nearest 10. You should calculate the value to the nearest 1.

   (iii)   (*2 points*) Using Euler's forward method, with a step size of $h = 0.25$, calculate $_{5.25}V_{65}^{(1)}$.

(e)   (*1 point*) The CCRC introduces a Full Life Care contract option, under which residence fees are level throughout their residency, regardless of the accommodation type. Entry fees are the same for both contract options.
State with reasons whether an individual would pay a higher residence fee while in ILU under the Full Life Care contract or the Modified Life Care contract.

5.

(*9 points*) A company has a defined contribution (DC) pension plan with the following features:

(i)   The company matches the employee's contributions to their DC fund, up to 4% of the employee's salary. The employee may contribute more than this, but these additional contributions are not matched.

(ii)   The employee's salary is paid at the end of each month; employee and employer contributions to the fund are made at this time.

(iii)   The investment returns on the fund are $i^{(12)} = 0.06$.

You are given the following information regarding an employee.

(i)   The employee entered the plan at age 25.

(ii)   Their starting salary was 60,000 per year.

(iii)   The employee is currently age 45.

(iv)   The employee has received annual pay increases of 3% at the start of each subsequent year of employment.

(v)   The employee has contributed 7% of their salary to their DC fund since entering the plan.

(a)   (*3 points*) Show that the accumulated amount in the DC fund (including the employee and employer contributions which have just been paid) is 322,000, to the nearest 1000. You should calculate the value to the nearest 50.

5. Continued

The employee is reviewing their retirement preparation, based on the following assumptions.

(i) They will continue to receive annual salary increases of 3%.

(ii) They will retire at age 65.

(iii) They will receive a benefit of 18,000 at the start of each year from their government's retirement program, starting at age 65.

(iv) At age 65 they will use the accumulated amount in their DC fund to purchase a whole life annuity due, payable annually.

(v) The individual is targeting a replacement ratio of 65%, including both the government and annuity benefits.

(b) *(2 points)* Show that the yearly annuity payment required to meet the target replacement ratio is 105,500 to the nearest 100. You should calculate the value to the nearest 1.

(c) *(3 points)* You are given that future investment returns will be $i^{(12)} = 0.06$, and that $\ddot{a}_{65} = 13.600$.

Calculate the level contribution rate needed by the employee over the next 20 years to accumulate sufficient funds to purchase this annuity,

(d) *(1 point)* Suppose that the employee wishes to use the fund to purchase a life-and-10-year-guaranteed annuity due instead of a whole life annuity due. State with reasons whether their replacement ratio will increase, decrease, or stay the same.

6.

(*10 points*)

(a)   (*2 points*) In the context of a Variable Annuity policy, describe the difference between a Guaranteed Minimum Maturity Benefit (GMMB) and a Guaranteed Minimum Income Benefit (GMIB).

An insurer issues a 10-year Variable Annuity with a single premium of 10,000 to (50). You are given:

> (i)   The GMMB is 90% of the single premium. There are no other guarantees.
>
> (ii)   The initial expense charge is 6% of the premium.
>
> (iii)   The management charge is 0.2% of the policyholder's fund, deducted at the end of each month, including at time 10.
>
> (iv)   The underlying fund asset value follows a geometric Brownian motion, with volatility $\sigma = 0.25$.
>
> (v)   The risk-free rate is $r = 4\%$ per year, compounded continuously.
>
> (vi)   There are no exits other than death.
>
> (vii)   Mortality is assumed to follow the Standard Ultimate Life Table.

(b)   (*4 points*) The insurer uses the Black-Scholes formula to construct a hedge portfolio of 10-year zero-coupon bonds and the underlying fund assets.

> (i)   (*3 points*) Show that the value of the hedge portfolio at time 0 is 1400 to the nearest 100. You should calculate the value to the nearest 1.
>
> (ii)   (*1 point*) Write down the value of the bond portion of the hedge portfolio at time 0.

(c)   (*4 points*) The insurer does not rebalance the hedge portfolio until time 1. At that time the policy is still in force and the price of the underlying fund assets has increased by 5%.

>   Calculate the rebalancing cost for the hedge portfolio.

END OF EXAMINATION