SOCIETY OF ACTUARIES

QFI Quantitative Finance Exam

Exam QFIQF

Date: Thursday, April 28, 2022

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 15 questions numbered 1 through 15 with a total of 100 points.

   The points for each question are indicated at the beginning of the question.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

3. Each question part or subpart should be answered either in the Word document, Excel document, or the paper provided as directed. Graders will only look at the work as indicated.

4. In the Word document, answers should be entered in the box marked ANSWER. The box will expand as lines of text are added. There is no need to use special characters or subscripts (though they may be used). For example, β_1 can be typed as beta_1 (and ^ used to indicate a superscript).

5. Prior to uploading your Word and Excel files, each file should be saved and renamed with your five-digit candidate number in the filename.

6. The Word file and the Excel file that contain your answers must be uploaded before time expires.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

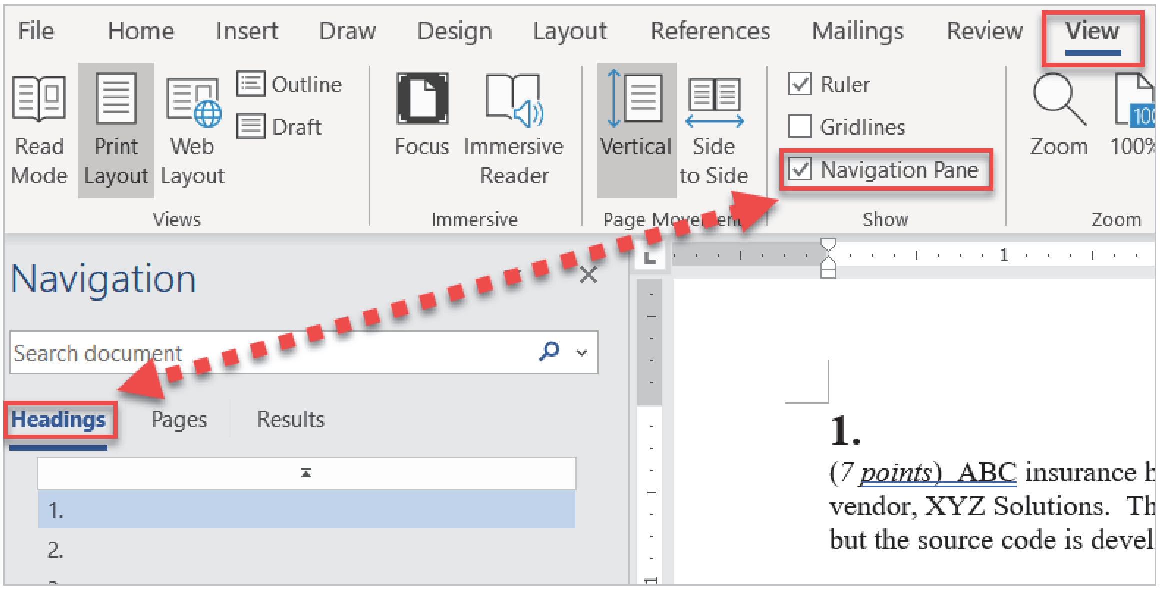
4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, hand in all your written-answer sheets to the Prometric center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

Navigation Instructions

Open the Navigation Pane to jump to questions.

Press Ctrl+F, or click View > Navigation Pane:



The responses for all parts of this question are required on the paper provided to you.

1.

(*9 points*) Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space and let $\{W_i(t)\}_{t\geq 0}$, $i = 1,2$, be two standard Brownian Motions with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Suppose that *r* is the continuously compounded constant risk-free rate and that S_1 and S_2 are two non-dividend paying risky assets whose dynamics under the risk-neutral measure $\mathbb{Q}$ are given by:

$$dS_i(t) = rS_i(t)dt + \sigma_i S_i(t)dW_i(t)\,, i = 1,2$$
$$dW_1(t)\,dW_2(t) = \rho dt$$

with $|\rho| < 1$. Moreover, define a new stochastic process $\{Z(t)\}_{t\geq 0}$ by:

$$Z(t) = \frac{1}{\sqrt{1-\rho^2}}\big(W_1(t) - \rho W_2(t)\big).$$

(a) (*1 point*) Compute $E^{\mathbb{Q}}\big(Z(t)\big)$ and $Var^{\mathbb{Q}}\big(Z(t)\big)$.

(b) (*2.5 points*) Show that $E^{\mathbb{Q}}\big(e^{sZ(t)}\big) = e^{\frac{s^2}{2}t}$ using Ito's Lemma.

(c) (*0.5 points*) Show that $Z(t)$ is uncorrelated with $W_2(t)$.

You may assume from this point on that $\{Z(t)\}_{t\geq 0}$ is a standard Brownian Motion with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let $Y(t) = S_1(t)/S_2(t)$, $0 \leq t \leq T$.

(d) (*2 points*) Derive the dynamics of $Y(t)$ in terms of $Z(t)$ and $W_2(t)$.

It can be shown that the diffusion term of the stochastic differential equation of $Y(t)$ can be written compactly as:

$$\sigma Y(t)dW(t)$$

with $\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$ and $W(t)$ another standard Brownian Motion with respect to the same filtration.

Let $\mathbb{P}$ be an equivalent probability measure to $\mathbb{Q}$, under which $Y(t), 0 \leq t \leq T$, is a martingale.

(e) (*1 point*) Derive the Radon-Nikodym derivative $\frac{d\mathbb{P}}{d\mathbb{Q}}$.

1. Continued

Recall that the payoff c_T of an exchange option between the two risky assets S_1 and S_2 at terminal time T is given as:

$$c_T = \ max\{S_1(T) - S_2(T), 0\}.$$

(f) (*2 points*) Derive today's price of the exchange option using your knowledge of the Black-Scholes formula and the probability measure $\mathbb{P}$.

The responses for all parts of this question are required on the paper provided to you.

2.

(*8 points*) Assume the non-dividend paying stock price S_t follows a Geometric Brownian Motion with constant volatility $\sigma = 10\%$. Let the continuously compounded risk-free rate of the market be $r = 2\%$. All expectations are under the risk-neutral measure $\mathbb{Q}$.

(a) (*1 point*) Prove that the discounted stock price $e^{-rt}S_t$ is a martingale.

Consider a special European-style option on S_t that expires at time $t = 5$ years. Let V_t represent the option price at time t, for $t \le 5$. At expiry, the payoff for this option is $V_5 = min\{S_3, S_5\}$.

(b) (*3 points*) Show that:

(i) $V_5 = S_5 \mathbb{I}_{\{S_3 \ge S_5\}} + S_3 \mathbb{I}_{\{S_3 < S_5\}}$ where $\mathbb{I}_{\{A\}} = \begin{cases} 1 \text{ if } A \text{ is true} \\ 0 \text{ if } A \text{ is false} \end{cases}$.

(ii) $P[S_3 < S_5] = 0.583$ under $\mathbb{Q}$ measure.

Assume from this point on that $t < 3$.

(c) (*2 points*) Show that:

(i) $E_t\left[S_3 \mathbb{I}_{\{S_3 < S_5\}}\right] = 0.619\, e^{-0.02t} S_t$.

(ii) $E_t\left[S_5 \mathbb{I}_{\{S_3 \ge S_5\}}\right] = 1.03\, E_t[S_3] E\left[e^{\sqrt{0.02}Z} \mathbb{I}_{\{Z \le -0.21\}}\right]$ with Z a standard normal random variable.

After further work, you have determined that:

$$E_t\left[S_5 \mathbb{I}_{\{S_3 \ge S_5\}}\right] = 0.401\, e^{-0.02t} S_t.$$

(d) (*1 point*) Calculate V_t and its Delta.

Your coworker claims that the special European-style option considered above can be Delta- and Gamma-hedged till its expiration by using a suitable short position in the underlying asset only.

(e) (*1 point*) Critique your coworker's claim.

The responses for all parts of this question are required on the paper provided to you.

3.

(*7 points*) Let $\{B_t : 0 \leq t \leq T\}$ be a standard Brownian Motion with respect to the filtration $\{I_t : 0 \leq t \leq T\}$, where $T > 0$ is some future date.

(a) (*1 point*) Define the Ito integral $\int_0^T B_t \, dB_t$ as the mean square limit of a suitable finite sum.

(b) (*3.5 points*)

(i) Define the quadratic variation $[B]_T$ of $\{B_t : 0 \leq t \leq T\}$ as the mean square limit of a suitable finite sum.

(ii) Show that $[B]_T = T$ using the definition in part (b) (i).

(iii) Interpret $(dB_t)^2 = dt$.

(c) (*1.5 points*) Compute $\int_0^T B_t \, dB_t$ using the definition in part (a).

(d) (*1 point*) Describe the behavior in terms of Brownian Motion trajectories of:

(i) The first-order variation.

(ii) Variations of order higher than two.

The responses for all parts of this question are required in the Excel spreadsheet.

4.

(*7 points*) Your company has a portfolio with two sets of installment payments receivable:

Installment payments 1: 2-year annual payment of $200,000 each
Installment payments 2: 5-year annual payment of $300,000 each.

(a) (*1 point*) Calculate the duration and the convexity of the portfolio, assuming the term structure of interest rates currently is flat at a continuously compounded rate of 2%.

The response for this part is to be provided in the Excel spreadsheet.

Your company's CFO suggests a Delta hedging strategy (duration matching) of using a 5-year zero-coupon bond to hedge against value changes of the portfolio from changes in interest rates.

(b) (*0.5 points*) Construct a hedging portfolio based on the CFO's suggestion.

The response for this part is to be provided in the Excel spreadsheet.

(c) (*1 point*) Calculate the estimated portfolio value changes using the duration-convexity approximation with and without the Delta hedging when interest rates increase by 10bps, 50bps and 100bps respectively.

The response for this part is to be provided in the Excel spreadsheet.

4. Continued

You recommend an alternative Delta-Gamma hedging strategy (duration-convexity matching) using two zero-coupon bonds (a 2-year zero-coupon bond and a 5-year zero-coupon bond).

(d) (*1 point*) Construct this alternative hedging portfolio using two zero-coupon bonds.

The response for this part is to be provided in the Excel spreadsheet.

(e) (*1.5 points*) Calculate the estimated portfolio value changes with the alternative hedging strategy you have recommended when interest rates increase by 10bps, 50bps and 100bps respectively.

The response for this part is to be provided in the Excel spreadsheet.

One analyst recommends a barbell-bullet bond portfolio to achieve positive portfolio returns. This analyst also claims that this barbell-bullet bond portfolio represents a short-term arbitrage opportunity if interest rates do not move significantly over time.

(f) (*1 point*) Explain how to construct a barbell-bullet bond portfolio.

The response for this part is to be provided in the Excel spreadsheet.

(g) (*0.5 points*) Explain whether the barbell-bullet bond portfolio can always achieve a positive portfolio return under small parallel shifts of interest rates.

The response for this part is to be provided in the Excel spreadsheet.

(h) (*0.5 points*) Critique the analyst's claims.

The response for this part is to be provided in the Excel spreadsheet.

The responses for all parts of this question are required in the Excel spreadsheet.

5.

(*5 points*) You are an actuarial analyst working at a financial institution. The financial institution has the following holdings in assets and liabilities. The current interest rate structure is flat at a continuously compounded rate of 2.0%.

Assets			Liabilities		
Item	Amount ($million)	Duration	Item	Amount ($million)	Duration
Cash	150	0	Deposits	550	0
S.T. Loans	350	0.9	S.T. Debt	380	0.4
M.T. Loans	580	3.5	M.T. Debt	320	4.5
L.T. Loans	620	11	L.T. Debt	150	9
Total	1,700		Total	1,400	

(a) (*1 point*) Calculate the dollar duration of the firm's equity.

The response for this part is to be provided in the Excel spreadsheet.

(b) (*0.5 points*) Explain implications of the duration mismatch for this firm.

The response for this part is to be provided in the Excel spreadsheet.

The firm is considering using interest rate swaps to stabilize the value of equity. Your team is asked to put together a proposal using a 5-year semi-annual swap at the current swap rate.

(c) (*0.5 points*) List two advantages of using swaps to protect against a decline in the value of the firm’s equity.

The response for this part is to be provided in the Excel spreadsheet.

5. Continued

As part of the analysis, your colleague already created the following table for a fixed semi-annual coupon bond with coupon rate at 2%.

T_i	Cash Flow CF	Discount Factor	Discounted CF	Weight	Weight * T_i	Weight * T_i^2
0.5	1	0.9900	0.99	0.010	0.0050	0.0025
1.0	1	0.9802	0.98	0.010	0.0098	0.0098
1.5	1	0.9704	0.97	0.010	0.0146	0.0218
2.0	1	0.9608	0.96	0.010	0.0192	0.0384
2.5	1	0.9512	0.95	0.010	0.0238	0.0595
3.0	1	0.9418	0.94	0.009	0.0283	0.0848
3.5	1	0.9324	0.93	0.009	0.0326	0.1143
4.0	1	0.9231	0.92	0.009	0.0369	0.1478
4.5	1	0.9139	0.91	0.009	0.0411	0.1852
5.0	101	0.9048	91.39	0.914	4.5716	22.8580

(d) (*2 points*) Construct a hedging portfolio using a 5-year semi-annual swap.

The response for this part is to be provided in the Excel spreadsheet.

You are given the following 6-month LIBOR forward curve and discount factors.

T_i	6-month LIBOR	$Z(0,T_i)$
0	2.32%	1
0.5	1.48%	0.9885
1.0	1.06%	0.9813
1.5	2.49%	0.9761
2.0	1.80%	0.9641

(e) (*1 point*) Calculate the swap rate C for 2-year semi-annual swap given the information provided above.

The response for this part is to be provided in the Excel spreadsheet.

The responses for all parts of this question are required on the paper provided to you.

6.

(*5 points*) Your company uses Black's model with caplets' forward volatilities to price caps and is interested in offering more general options. You have chosen the LIBOR market model for pricing these options.

Given:

- $f_n(t, \tau, T)$ is the n-times compounded annual forward rate as seen at time t for the period $[\tau, T]$.
- $r_n(\tau, T)$ is the n-times compounded annual LIBOR spot rate at time τ for the period $[\tau, T)$.
- $\sigma_f^{fwd}(T_{i+1}), i = 0,1,2,3,4$ are the caplet volatilities given in the table below

i	T_i	$\sigma_f^{fwd}(T_{i+1})$
1	0.25	0.2
2	0.50	0.22
3	0.75	0.26
4	1	0.29

- Forward rate volatility σ_f^{i+1} of $f_n(t, T_i, T_{i+1})$ is as given below

$$\sigma_f^{i+1}(t) = \begin{cases} S_1, t < T_1, i = 0 \\ S_2, T_1 \leq t < T_2, i = 1 \\ S_3, T_2 \leq t < T_3, i = 2 \\ S_4, T_3 \leq t < T_4, i = 3 \end{cases}$$

(a) (*1 point*) Calculate the corresponding forward rate volatilities $S_i, i = 1,2, \ldots, 4$.

6. Continued

Consider a security with payoff $G(r_n(\tau, T))$ at time T given by

$$G(r_n(\tau, T)) = N\,\Delta\,(2r_n(\tau, T) - r_K),$$

where

- N is the notional amount
- $n = \frac{1}{\Delta}, \tau = T - \Delta$
- r_K is the strike rate.

(b) (*2 points*)

(i) Show that the price V at time 0 of the security is given by:

$$V = Z(0, T) N\,\Delta\,\{2f_n(0, \tau, T) - r_K\}.$$

Consider now the case in which the same payoff $N\,\Delta\,(2r_n(\tau, T) - r_K)$ is paid at τ instead at time T.

(ii) Show that, if the payoff is paid at time τ, the price V at time 0 of the security is given by:

$$V = Z(0, T) N\,\Delta\,\{2f_n(0, \tau, T) - r_K - f_n(0, \tau, T)\, r_K \Delta + 2E_f^*[r_n(\tau, T)^2]\Delta\},$$

where

- $Z(\tau, T)$ is the value of zero-coupon bond at time τ with maturity T.
- $f_n\,(0, \tau\,, T)$ is the n-times compounded forward rate at 0 for an investment at τ and maturity T.
- σ_f is the volatility of the forward rate implied from caplet prices.
- $E_f^*[r_n(\tau, T)]$ is the expected value of $r_n(\tau, T)$ with respect to T-forward dynamics.

6. Continued

You are provided the following information:

- Notional amount N is \$100 million.
- $\tau = 0.5$
- $T = 0.75$
- $Z(0, 0.5) = 0.989$
- $Z(0, 0.75) = 0.982$
- $\sigma_f(T) = 0.22$
- $r_K = 0.02$

(c) (*2 points*) Calculate the value of the security using the formula in part (b) (ii) and assuming the LMM.

7.

(*8 points*) The risk neutral dynamics of the short rate under the two-factor Vasicek model are as follows:

$$r_t = \phi_{1t} + \phi_{2t}$$
$$d\phi_{1t} = \gamma_1(\overline{\phi_1} - \phi_{1t})dt + \sigma_1 dW_{1t}$$
$$d\phi_{2t} = \gamma_2(\overline{\phi_2} - \phi_{2t})dt + \sigma_2 dW_{2t}$$

where $\gamma_1, \gamma_2, \sigma_1, \sigma_2, \overline{\phi_1}$, and $\overline{\phi_2}$ are constants, and W_{1t} and W_{2t} are independent standard Wiener processes.

(a) (*1.5 points*) Derive a formula for r_t in terms of W_{1t} and W_{2t}.

The response for this part is required on the paper provided to you.

Recall that a one-factor Vasicek model can be represented by just $r_t = \phi_{1t}$, and that the time *t* price of a zero-coupon bond maturing at time *T* is $Z(t,T) = e^{A(t,T)-B_1(t,T)\phi_{1t}}$, where

$$B_1(t,T) = \frac{1}{\gamma_1}(1 - e^{-\gamma_1(T-t)})$$
$$A(t,T) = (B_1(t,T) - (T-t))\left(\overline{\phi_1} - \frac{\sigma_1^2}{2\gamma_1^2}\right) - \frac{\sigma_1^2 B_1(t,T)^2}{4\gamma_1}$$

(b) (*1.5 points*) Derive $Z(t,T)$ under the two-factor Vasicek model.

The response for this part is required on the paper provided to you.

7. **Continued**

(c) (*4 points*) Perform each of the following calculations respectively for the one-factor and the two-factor Vasicek models.

(i) Derive the $Correlation(r_t(T-t), r_t)$ where $r_t(T-t) = -\frac{\log(Z(t,T))}{T-t}$.

The response for this part is required on the paper provided to you.

You are given the following parameter values for the two models for part (ii):

	Two Factor Model		One Factor Model
	$i = 1$	$i = 2$	$i = 1$
γ_i	0.6	-0.1	0.2522
$\bar{\phi}_i$	0.01	0	0.04
σ_i	0.02	0.01	0.0224
ϕ_{i0}	-1%	1.5%	0.5%

(ii) Calculate $Z(15)$ at current t = 0.

The response for this part is to be provided in the Excel spreadsheet.

You are given the following two scenarios for the values of ϕ_{i0} for part (iii)

	Two Factor Model		One Factor Model
	$i = 1$	$i = 2$	$i = 1$
ϕ_{i0} scenario 1	-1%	1.5%	0.5%
ϕ_{i0} scenario 2	0%	1.5%	1.5%

(iii) Graph annualized yields of zero-coupon bonds under the two scenarios in the same graph against maturity for $T \leq 20$.

The response for this part is to be provided in the Excel spreadsheet.

(iv) Describe one advantage of the two-factor Vasicek model over the one-factor version observed in parts (c)(i) and (c)(iii).

The response for this part is required on the paper provided to you.

7. Continued

(d) (*1 point*) Describe one additional advantage of the two-factor Vasicek model over the one-factor version regarding the volatility of $dr_t(T - t)$.

The response for this part is required on the paper provided to you.

The responses for all parts of this question are required on the paper provided to you.

8.

(*6 points*) You are an investment actuary at an insurance company. You are asked to improve the interest rate model of your company to reflect a negative interest rate environment.

Currently, your company uses the LIBOR Market Model (LMM) that models forward and zero-coupon rates in a strictly positive fashion. The calibration is being done with the use of observable LIBOR-linked assets data.

To adopt the negative rates, you suggested the inclusion of a shift parameter in the LIBOR Market Model. However, the Chief Actuary is concerned that this approach would complicate the calibration process.

(a) (*1 point*) Explain whether you agree or disagree with the Chief Actuary's concern.

You recommended two shifted LMM models:

- Displaced Diffusion LMM (DD-LMM), and
- LMM+

(b) (*1 point*) Identify a key difference between these two models.

(c) (*1 point*) Explain the implication of a low shift parameter to the interest rate model.

To prepare for the market transition away from LIBOR benchmark, you were also asked to review the construction of term benchmark rate from overnight risk-free rates (RFRs) using the following two methodologies.

- Backward-looking
- Forward-looking

(d) (*1.5 points*) Compare and contrast the two methodologies.

In addition, you are asked to construct the interest rates term structure from derivatives linked to the new RFRs.

(e) (*1 point*) Explain how the term structure constructed from derivatives linked to the new risk-free rates would be different from the ones derived from LIBOR.

8. **Continued**

Your company is trying to develop an asset liability management strategy using assets benchmarked to the new overnight risk-free rate benchmarks.

(f) (*0.5 points*) Explain the shortcoming of this strategy and recommend a solution for this.

The responses for all parts of this question are required on the paper provided to you.

9.

(*4 points*) You are evaluating the domestic and a foreign bond market. The domestic bond price is given by $P^D(t,T)$ and the foreign bond price is given by $P^F(t,T)$. The domestic forward rate $f^D(t,T)$ and the foreign forward rate $f^F(t,T)$ follow the dynamics:

$$df^D(t,T) = \alpha^D(t,T)dt + \sigma^D(t,T)dW^D(t)$$

$$df^F(t,T) = \alpha^F(t,T)dt + \sigma^F(t,T)dW^F(t)$$

where, W^D is a Brownian Motion under the domestic martingale measure and W^F is a Brownian Motion under the foreign martingale measure. The exchange rate X (in units of domestic currency per unit of foreign currency) follows the dynamics:

$$dX_t = \mu(t)X_t dt + X_t \sigma_X(t)dW^X(t)$$

(a) (*1.5 points*) Show that the short rate $r^D(t)$ is normally distributed when $\sigma^D(t,T)$ is deterministic.

Assume that the foreign and domestic market prices of risk for forward rates follow the relationships:

$$\lambda^F(t) \equiv \frac{\mu^F(t,T) - \alpha^F(t,T)}{\sigma^F(t,T)},$$

$$\lambda^D(t) \equiv \frac{\mu^D(t,T) - \alpha^D(t,T)}{\sigma^D(t,T)},$$

$$\lambda^F = \lambda^D - \sigma_X.$$

(b) (*2 points*) Show that under the domestic martingale measure the foreign forward rate drift is:

$$\tilde{\alpha}^F(t,T) = \sigma^F(t,T)\left(\int_t^T \sigma^F(t,s)ds - \sigma_X(t)\right)$$

(c) (*0.5 points*) Determine the condition under which domestic and foreign martingale measures are equivalent.

The responses for all parts of this question are required in the Excel spreadsheet.

10.

(*7 points*) Assume the risk-free rate is zero and the current level of the S&P 500 is 4,000. The market prices of one-year options on the S&P 500 are listed in the following table:

Strike K(i)	Call C(i)	Put P(i)
3200	847.44	47.44
3400	686.45	86.45
3600	543.56	143.56
3800	420.78	220.78
4000	318.62	318.62
4200	236.22	436.22
4400	171.68	571.68
4600	122.48	722.48
4800	85.89	885.89

You want to estimate the market price of a one-year variance swap on the S&P 500 using the piecewise-linear replication strategy.

(a) (*2 points*)

(i) Write down the piecewise-linear replication function to approximate the payoff at expiration.

(ii) Explain the key parameters used in the function.

The response for this part is to be provided in the Excel spreadsheet.

(b) (*2 points*) Describe the key steps in the piecewise-linear replication strategy.

The response for this part is to be provided in the Excel spreadsheet.

(c) (*2 points*) Estimate the market price of the one-year variance swap on the S&P 500.

The response for this part is to be provided in the Excel spreadsheet.

(d) (*1 point*) Assess without calculations whether the piecewise linear approximation underestimates or overestimates the value of the variance swap.

The response for this part is to be provided in the Excel spreadsheet.

The responses for all parts of this question are required on the paper provided to you.

11.

(*6 points*) Consider a European option on a non-dividend paying stock. Assume that

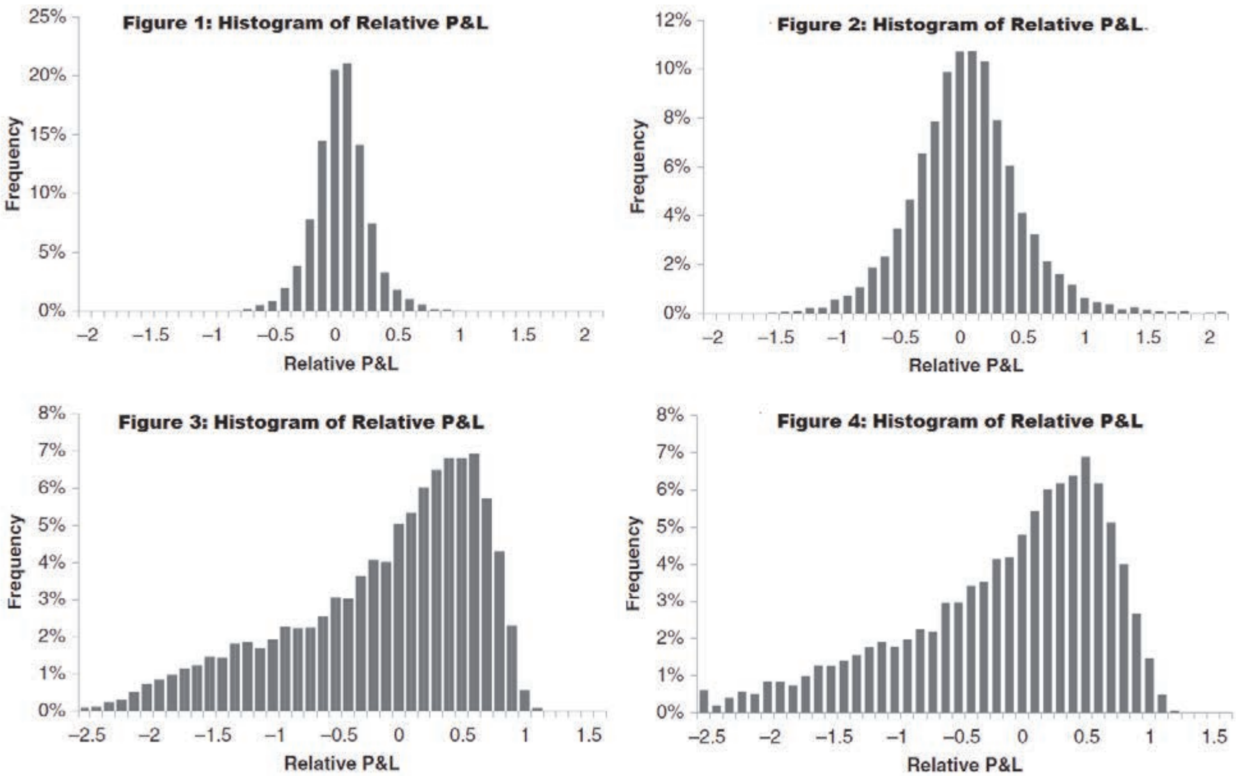
- The stock price *S* follows Geometric Brownian Motion:
  $$dS = \mu S dt + \sigma S dW, \ \ where\ \mu > 0,\ \sigma = 20\%.$$
- There are no transaction costs.

You are given the following four delta hedging strategies on the option:

Strategy 1: Use 20% volatility to determine Delta and rebalance hedge position daily.
Strategy 2: Use 40% volatility to determine Delta and rebalance hedge position daily.
Strategy 3: Use 20% volatility to determine Delta and rebalance hedge position weekly.
Strategy 4: Use 40% volatility to determine Delta and rebalance hedge position weekly.

11. **Continued**

According to the stock price process, your analyst ran Monte Carlo simulations and produced the following histograms of Relative P&L of each strategy, where Relative P&L refers to the Profit/Loss that is measured relative to what would be the Black-Scholes-Merton (BSM) fair value of the option if you replicated continuously at the realized volatility. That is, Relative P&L = Present Value of Payoff – BSM Fair Value.



11. **Continued**

(a) (*1.5 points*) Explain which Strategy is associated with Figure 1 and Figure 2, respectively.

(b) (*1.5 points*) Explain why Figure 3 looks similar to Figure 4.

Assume that transaction cost is proportional to the value of the stocks traded.

(c) (*1.5 points*)

(i) Sketch the histograms of relative P&L for Strategy 1 and Strategy 3, respectively. Note: You need not mark any values on your *x*-axis and *y*-axis. The key is to show the shape or contour of the histogram.

(ii) Explain the key drivers for the differences in the histogram.

Let m_1 and m_3 be the mean value of relative P&L of Strategy 1 and Strategy 3, respectively.

Let s_1 and s_3 be the standard deviations of relative P&L of Strategy 1 and Strategy 3, respectively.

(d) (*1 point*) Compare m_1 vs. m_3 vs. 0. Justify your ranking.

(e) (*0.5 points*) Compare s_1 vs. s_3. Justify your ranking.

The responses for all parts of this question are required on the paper provided to you.

12

(*6 points*) Assume the Black-Scholes (B-S) framework.

The current B-S price of a T-year European call option with strike price K on a non-dividend paying stock is:

$$C = SN(d_1) - Ke^{-rT}N(d_2)$$

where:

$N(\cdot) = \text{cumulative normal distribution}$
$S = \text{the current price of the stock}$
$r = \text{the continuously compounded risk-free interest rate}$
$\sigma = \text{the volatility of the stock's continuously compounded returns}$

$$d_1 = \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln\frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

You are also given that:

$$SN'(d_1) = Ke^{-rT}N'(d_2)$$

(a) (*1.5 points*) Show that the Gamma of the European call is:

$$\text{Gamma} = N'(d_1)\frac{1}{S\sigma\sqrt{T}}$$

(b) (*0.5 points*) Prove that the Gamma of a European call is equal to the Gamma of an otherwise equivalent European put.

12. Continued

(c) (*2 points*) Identify whether each of the following statements is true or false. Briefly justify your answer.

(i) Gamma approaches 0 for deep-in-the money calls.

(ii) Gamma approaches 1 for deep-out-of-the-money puts.

(iii) For an out-of-the-money option with an underlying asset price that is exhibiting low volatility, Gamma is expected to be relatively low.

(iv) For an option that happens to be right at-the-money very near to the expiry date, a stable Gamma is likely to be observed.

The table below lists the Greek measures of two long European options on Stock XYZ.

Option	Delta	Gamma	Theta (per day)
A	0.7890	0.0180	-0.0118
B	0.6795	0.0222	-0.0129

Your company has a portfolio consisting of a long position in 100 Option A and a short position in 200 Option B.

(d) (*1.5 points*) Estimate the change in the value of your company's portfolio by using the Taylor series expansion of the option prices if the price of the underlying stock decreases by 10 after 5 days.

Your co-worker comments that the B-S model is not robust because one of the assumptions underpinning the model is that hedging is continuous and thus the B-S model does not apply if hedging is discrete.

(e) (*0.5 points*) Critique your co-worker's comment.

The responses for all parts of this question are required on the paper provided to you.

13.

(*5 points*) You would like to create a Monte Carlo simulation of a three-day barrier option using forward-starting volatilities. The at-the-money volatility structure for the spot-based asset is as follows.

Tenor	Market Volatility
1 day	19%
2 day	20%
3 day	18%

(a) (*1 point*) Calculate the Monte Carlo step volatility of the barrier option for each of the three days.

Your colleague is considering replacing the current model with a stochastic volatility model.

(b) (*1 point*) Describe three limitations of stochastic volatility models.

To overcome the limitations of stochastic volatility models, a colleague suggests modeling using a local stochastic volatility model.

(c) (*1 point*) Describe three limitations of local stochastic volatility models.

Your colleague decides to model volatility V using the following stochastic process:

$$\frac{dV}{V} = \alpha\, dt + \xi\, dW$$

where α and ξ are positive constants and W is a standard Brownian Motion.

(d) (*1 point*) Critique this choice of model for volatility and, if appropriate, suggest a better model.

13. **Continued**

Assume that volatility can be described by the following mean-reverting discrete time series model:

$$\Delta\sigma_t = \sigma_{t+1} - \sigma_t = 0.3(15\% - \sigma_t) + \epsilon_t$$

where ϵ_t is a random variable with zero mean, $\epsilon_0 = 3\%$ and $\epsilon_1 = -2\%$. The initial volatility σ_0 is 16%.

(e) (*1 point*) Determine the value of σ_2.

The responses for all parts of this question are required in the Excel spreadsheet.

14.

(*8 points*) Volco VA company (VVA) is looking to manage the volatility associated with its Variable Annuity (VA) and Equity-Indexed Annuity (EIA) businesses. One of the fund managers suggested that the company consider managed-volatility funds as part of its new product line. He has asked the actuarial area to analyze the feasibility of these fund types, with a particular interest in target and capped volatility funds.

The fund manager is interested in the performance of a target volatility fund under a market scenario. He has asked an analyst to generate the following equity prices and forward volatilities for 4 years (Table 1), so that you can calculate the corresponding fund returns.

Table 1	Years (t)			
	0	1	2	3
S_t	100	88	105	110
σ_t	20%	40%	10%	30%

Table 2	Years (t)			
	0	1	2	3
Equity %	75%	α_1	α_2	α_3
Bond %	25%	1 -α_1	1 -α_2	1 -α_3
Equity Price	100	88	105	110
Bond Index Price	100	103.05	106.18	109.42
Target Vol Fund Price	100	X	100.16	Y

(a) (*1.5 points*) Calculate the resulting target volatility fund prices X and Y in Table 2, assuming a continuously compounded risk-free rate of 3%, a target volatility of 15% and a maximum equity % of 200%.

The response for this part is to be provided in the Excel spreadsheet.

14. Continued

The fund manager is concerned about how the fund volatility will perform under both high and low volatility scenarios.

Table 3	
Stock Returns	Realized Volatility
Negative	$>\sigma_C$
Positive	$= \sigma_T$
Negative	$<\sigma_T$

(b) (*1.5 points*) Compare the relative performance of the target volatility fund, capped volatility fund, and underlying asset under the scenarios in Table 3, where the target volatility = σ_T and cap volatility = σ_C and $\sigma_T < \sigma_C$.

The response for this part is to be provided in the Excel spreadsheet.

As part of the analysis, the Hedging Actuary has asked you to perform an investigation of the properties of the target volatility fund.

(c) (*1 point*) Explain whether the following statements are True or False:

(i) Call options on a target volatility fund should be cheaper than or equal to the equivalent call options on the underlying risky-asset.

(ii) Call options on a capped volatility fund should be cheaper than or equal to the equivalent call options on the underlying risky-asset.

The response for this part is to be provided in the Excel spreadsheet.

As an alternative to changing the fund offering, the CRO has proposed the possibility of changing its maturity guarantee product with a target volatility fund to a product with a capped volatility fund.

(d) (*1 point*) Compare the impact on the company's market risk of offering a capped volatility fund versus the target volatility fund in the product design.

The response for this part is to be provided in the Excel spreadsheet.

14. **Continued**

In addition to VA, the insurance company would like to introduce an EIA product with a participation rate on the equity returns and a guaranteed payoff of 100% of initial deposit at the 1-year maturity (Note: no cap on the credited rate). Suppose a policyholder wishes to deposit $10,000 into the EIA product, where the underlying fund of the product is an equity index fund with volatility of 25%. Assume the continuously compounded risk-free rate is 2% and the current equity index fund's price is 100.

(e) (*1.5 points*)

(i) Calculate the risk budget of the EIA product.

(ii) Calculate the number of at-the-money call options that may be purchased using the risk budget in part (e)(i).

(iii) Calculate the break-even participation rate that can be funded with the risk budget in part (e)(i).

The response for this part is to be provided in the Excel spreadsheet.

Suppose the company is also considering whether to offer the EIA product with an underlying fund that has a target volatility of 15%.

(f) (*1.5 points*)

(i) Calculate the break-even participation rate that can be funded with the risk budget in part (e)(i).

(ii) Recommend whether to switch to an underlying fund with a target volatility.

The response for this part is to be provided in the Excel spreadsheet.

The responses for all parts of this question are required in the Excel spreadsheet.

15.

(*9 points*) You are an actuary for an insurer that offers a structured product based variable annuity (spVA) with a buffered and capped payout.

An investor invested an initial premium of \$500,000 in a spVA product of 3-year segment linked to the stock index S starting on January 3, 2017 with a buffer level of 20% and a cap level of 25%. This product has the following payout structure:

$$payout(S_3) = \begin{cases} \min(100{,}000 + S_3, 500{,}000), & S_3 < 500{,}000 \\ S_3, & 500{,}000 \leq S_3 < 625{,}000 \\ 625{,}000, & S_3 \geq 625{,}000 \end{cases}$$

where S_3 is the final Account Value at the maturity which is determined by the initial premium times (1 + stock index return over 3 years).

The Interim Account Value (AV) is the lesser of the calculated value from the pro-rated cap and the value of the underlying bonds and options positions. The buffer is not pro-rated.

Suppose that

- the stock index S is 207.5 on January 3, 2017
- the stock index S is 254 on January 3, 2018.
- The annual at-the-money implied volatility is 20%.
- The annual risk-free rate is 3% (continuously compounded).
- The annual dividend yield is 2% (continuously compounded).

(a) (*3 points*)

(i) (*0.5 points*) Calculate the pro-rated cap value component of the Interim AV.

(ii) (*2.5 points*) Specify and justify a portfolio of bonds and options that provides the maturity payout of the spVA product (for each instrument specify whether it is a long or short position, the strike price, bond principal, bond coupon, and the time-to-maturity).

The response for this part is to be provided in the Excel spreadsheet.

15. Continued

Your manager believes that it is more accurate to incorporate volatility skew into the model when valuing the underlying options position of the spVA product. Based on the market data, your manager has fitted an implied volatility function $\Sigma(K) = 20\% - .05\%*(K - 254)$, where K is the strike level.

(b) (*1 point*) Assess your manager's approach.

The response for this part is to be provided in the Excel spreadsheet.

Your junior assistant made the following statement:

> "Using this modeling approach will also impact our delta hedging strategy for this product"

(c) (*0.5 points*) Critique his statement.

The response for this part is to be provided in the Excel spreadsheet.

(d) (*3.5 points*) Calculate the Interim AV for the policy as of January 3, 2018, using

(i) (*2 points*) The fitted implied volatility function provided by your manager.

(ii) (*1.5 points*) The implied volatility function evaluated only at K = 207.5, namely $\Sigma(K=207.5) = 20\% - .05\%*(207.5 - 254)$.

The response for this part is to be provided in the Excel spreadsheet.

(e) (*1 point*) Evaluate whether including volatility skew will lead to lower or higher interim AV based on the results in part (d) above.

The response for this part is to be provided in the Excel spreadsheet.

****END OF EXAMINATION****