SOCIETY OF ACTUARIES

QFI Quantitative Finance Exam

Exam QFIQF

Date: Thursday, October 27, 2022

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 15 questions numbered 1 through 15 with a total of 100 points.

   The points for each question are indicated at the beginning of the question.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

3. Each question part or subpart should be answered either in the Excel document or the paper provided as directed. Graders will only look at the work as indicated.

4. In the Excel document, answers should be entered in the box marked ANSWER. The box will expand as lines of text are added. There is no need to use special characters or subscripts (though they may be used). For example, β_1 can be typed as beta_1 (and ^ used to indicate a superscript).

5. Prior to uploading your Excel files, each file should be saved and renamed with your five-digit candidate number in the filename.

6. The Excel file that contain your answers must be uploaded before the five-minute upload period expires.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, hand in all your written-answer sheets to the Prometric center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

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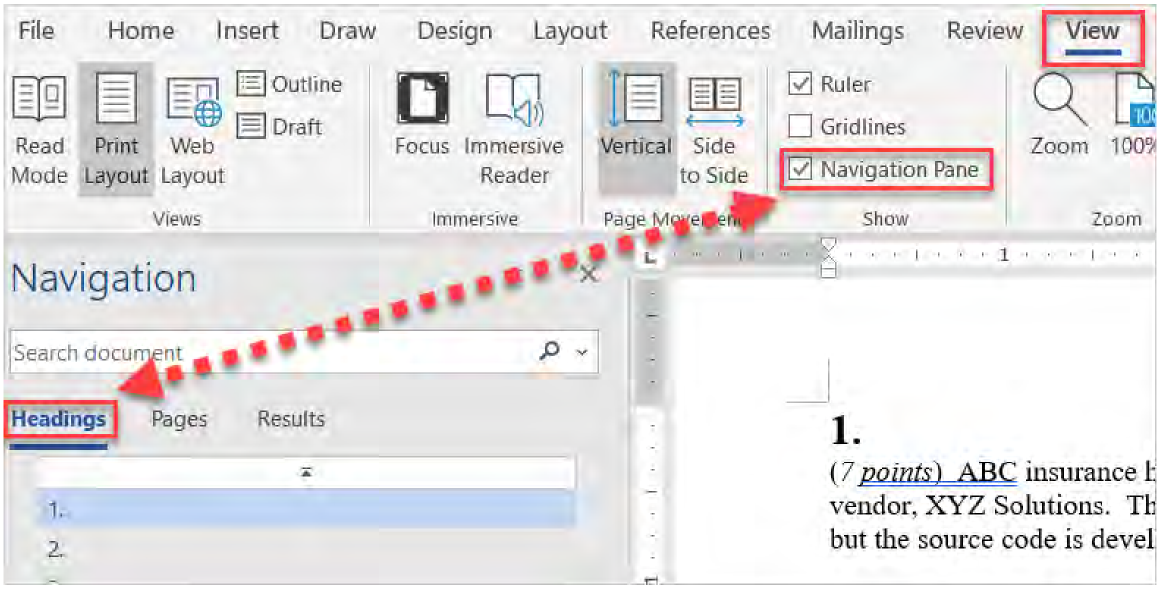
Navigation Instructions

Open the Navigation Pane to jump to questions.

Press Ctrl+F, or click View > Navigation Pane:



The responses for all parts of this question are required on the paper provided to you.

1.

(*5 points*) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{W_t\}_{t\geq 0}$ be a standard Wiener process with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.
Also assume a constant risk-free rate $r > 0$.

(a) (*1 point*) List the criteria for a stochastic process to be a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Let $X_t = \int_0^t W_s{}^2 ds + \alpha_t$, where $\alpha_t = f(W_t, t)$.

(b) (*0.5 points*) Derive a stochastic differential equation for X_t using Ito's Lemma.

Assume from this point on that $\alpha_t = -tW_t{}^2 + \beta_t$, with β_t a deterministic function of time.

(c) (*1.5 points*) Identify an appropriate β_t, if it exists, that makes X_t a martingale.

(d) (*2 points*) Calculate $E(W_t{}^4)$ using Ito's Lemma.

The responses for all parts of this question are required on the paper provided to you.

2.

(*6 points*) You are considering investing in an asset S that is known to attain the following discrete prices:

$$S_0 = 100,$$

$$S_t = \begin{cases} 1.3S_{t-1} \text{ with probability } 0.7 - t/20 \\ \alpha S_{t-1} \text{ with probability } 0.3 + t/20 \end{cases} \qquad \text{(for } t = 1, 2, \dots, 10\text{)},$$

where α is a positive constant.

Assume that the above probabilities are under the real-world measure and the annual effective risk-free rate of interest r is 5%.

(a) (*0.5 points*) Determine the range of α so that there is no arbitrage opportunity.

Let d be the number of annual down-movements for $t \leq 4$.

(b) (*0.5 points*) Derive the price S_4 as a function of d and give the possible range of S_4.

Let $\alpha = 0.9$ and consider a double barrier option Z that pays 100 at $t = 5$ if S_t either exceeds 290 or falls below 70 at any time $t \leq 5$.

(c) (*1.5 points*) Calculate the real-world probability that the double barrier option will be exercised.

(d) (*0.5 points*) Calculate the risk-neutral probability of an up-movement in the price of the asset S.

(e) (*1 point*) Calculate the price Z_0 of the double barrier option at $t = 0$ under the risk-neutral measure.

Let a standard European call option C pay $\max\{0, S_5 - 110\}$ at $t = 5$.

(f) (*2 points*) Derive a lower bound for the price C_0 of the call option at $t = 0$ under the risk-neutral measure using Jensen's inequality.

The responses for all parts of this question are required on the paper provided to you.

3.

(*8 points*) You are reviewing the pricing of an equity-indexed annuity (EIA), which is an annuity with an embedded guarantee associated to a stock index plus a zero-coupon bond. You are given:

- The stock index has level $S(t)$ at time t and follows a stochastic differential equation of the form:

$$\frac{dS(t)}{S(t)} = r(t)\,dt + \sigma_S(t)\,dW_S(t).$$

- The zero-coupon bond price process $B(t,T)$ at time t with maturity time T satisfies the following stochastic differential equation:

$$\frac{dB(t,T)}{B(t,T)} = r(t)\,dt + \sigma_B(t,T)\,dW_B(t).$$

Assume that the standard Wiener processes $\{W_S(t)\}_{t\geq 0}$ and $\{W_B(t)\}_{t\geq 0}$ are correlated with constant correlation $\rho < 1$, defined under the same probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where $\mathbb{Q}$ is a risk-neutral measure. Moreover, $r(t)$, $\sigma_S(t)$ and $\sigma_B(t,T)$ are time-dependent deterministic functions, with $r(t)$ denoting the risk-free rate.

(a) (*1 point*) List two potential issues of using real-world probabilities to price an EIA.

(b) (*1 point*) Show that the discounted stock index process $e^{-\int_0^t r(u)du}S(t)$ is a martingale with respect to $\mathbb{Q}$.

Consider the forward price process $F(t) = \frac{S(t)}{B(t,T)},\ t < T$.

(c) (*2 points*) Show that for some deterministic function $\theta(t)$,

$$\frac{dF(t)}{F(t)} = \theta(t)dt + [\sigma_S(t)\,dW_S(t) - \sigma_B(t,T)\,dW_B(t)]$$

3. Continued

For the remainder of this question, you may assume that $\{W(t)\}_{t\geq 0}$ defined by:

$$W(t) = \int_0^t \frac{\sigma_S(u)}{\sqrt{\sigma_S(u)^2 - 2\rho\sigma_S(u)\sigma_B(u,T) + \sigma_B(u,T)^2}}\, dW_S(u)$$
$$-\int_0^t \frac{\sigma_B(u,T)}{\sqrt{\sigma_S(u)^2 - 2\rho\sigma_S(u)\sigma_B(u,T) + \sigma_B(u,T)^2}}\, dW_B(u)$$

is a standard Wiener process under $(\Omega, \mathcal{F}, \mathbb{Q})$.

(d) (*2.5 points*) Show that $\frac{dF(t)}{F(t)} = \sigma(t)\, dW^*(t)$ for suitably defined $\sigma(t), W^*(t)$, and probability measure $\mathbb{P}$ using Girsanov's theorem.

Your EIA product has a participation rate of 80% on the appreciation of the index fund, and annual minimum guaranteed rate of 1% with an annual cliquet crediting option.

For *s* > *t*, denote by *C(s, r, S)* the contingent payoff amount at discrete time *s*, where $r = r(s)$ and $S = S(s)$.

C_t is the time-*t* price of *C(s, r, S).*

(e) (*1.5 points*) Express C_t in terms of a risk-neutral expectation involving *r(t)* and *S(t).*

The responses for all parts of this question are required on the paper provided to you.

4.

(*7 points*) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{W_t\}_{t\geq 0}$ be a standard Wiener process.

(a) (*1.5 points*) Show that $Z_t = \int_0^t W_u du$ is a normally distributed random variable.

Define a stochastic process $\{Y_t\}_{t\geq 0}$ given by $Y_0 = 0$, $Y_t = \frac{\sqrt{3}}{t} Z_t$ for $t > 0$.

(b) (*3.5 points*) Determine whether Y_t is a Wiener process with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Suppose S_t satisfies the Geometric Brownian Motion model:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t,$$

with μ and σ both constant, and define a new stochastic process:

$$G_t = e^{\frac{1}{t}\int_0^t \ln(S_u)\, du}, G_0 = S_0.$$

(c) (*2 points*) Derive an expression for dG_t in terms of dY_t.

The response for this part is to be provided in the Excel spreadsheet.

5.

(*5 points*) Consider the following market conditions:

Time Period T_i	Discount Factor $z(0,T_i)$	Initial Spot Rate $r(0,T_i)$	Forward Rate
0	1		
1	0.99009901	0.01	
2	0.961168781	0.02	
3	0.915141659	0.03	
4	0.854804191	0.04	
5	0.783526116	0.05	

(a) (*0.5 points*) Calculate the current 1 year forward rates implied by the above initial spot rates.

(b) (*0.5 points*) Describe the cost to enter into a Forward Rate Agreement (FRA) today on \$100 million for one year starting 3 years into the future assuming no transaction costs.

Suppose now at time 1 that the spot rate curve has shifted down 25 basis points as indicated below:

Time Period T_i	Discount Factor $z(1,Ti)$	Spot Rate $r(1,Ti)$	Forward Rate
1	1		
2		0.0075	
3		0.0175	
4		0.0275	
5		0.0375	

(c) (*2 points*) Determine the loss or gain if you were to exit the FRA at time 1.

5. **Continued**

(d) (*1.5 points*) Discuss the advantages and disadvantages of using forwards rather than futures for each of the following risks:

(i) Basis Risk

(ii) Tailing of the hedge

(iii) Liquidity

(iv) Credit Risk

Assume that the management is concerned about a larger increase of rates beyond 25 basis points at time 3.

(e) (*0.5 points*) Describe one other instrument that may be more suitable to protect against such a situation.

6.

(*6 points*)

(a) (*1 point*) Express partial DV01s for *n* assets with respect to the forward rate variables in the Jacobian matrix multiplication form by perturbating the curve rather than perturbating the inputs.

The response for this part is required on the paper provided to you.

A portfolio *P* consists of 4 fixed income instruments:

(1) (I-1) \$100 notional one-year zero-coupon bond
(2) (I-2) \$100 notional two-year zero- coupon bond
(3) (I-3) \$100 notional 3-year annual coupon bond with coupon rate of 4%
(4) (I-4) \$100 notional 4-year annual coupon bond with coupon rate of 4.5%

	Instruments payments			
	year 1	year 2	year 3	year 4
I-1	100.00			
I-2		100.00		
I-3	4.00	4.00	104.00	
I-4	4.50	4.50	4.50	104.50

The current forward and spot rates are as follows:

	forward rates *f*	discount factors *d*	spot rates *s*
year 1	0.0300	0.9704	0.0300
year 2	0.0350	0.9371	0.0325
year 3	0.0400	0.9003	0.0350
year 4	0.0450	0.8607	0.0375

6. **Continued**

Assuming *dr* is a 1 basis point perturbation, here are four sets of perturbed spot rates and corresponding perturbed discount factors:

	perturbed spot rates			
	r_1	r_2	r_3	r_4
year 1	**0.0301**	0.0300	0.0300	0.0300
year 2	0.0325	**0.0326**	0.0325	0.0325
year 3	0.0350	0.0350	**0.0351**	0.0350
year 4	0.0375	0.0375	0.0375	**0.0376**

	perturbed discount factors			
	d_1	d_2	d_3	d_4
year 1	**0.9703**	0.9704	0.9704	0.9704
year 2	0.9371	**0.9369**	0.9371	0.9371
year 3	0.9003	0.9003	**0.9001**	0.9003
year 4	0.8607	0.8607	0.8607	**0.8604**

(b) (*4 points*) Calculate the partial DV01 of the portfolio *P* with respect to forward rates (f_1,f_2,f_3,f_4) based on the four sets of perturbed discount factors.

The response for this part is to be provided in the Excel spreadsheet.

Assume a set of dates up to maturity T_N are defined as $T_1, T_2, \cdots T_{N-1}, T_N$ with $T_i = T_{i-1} + \Delta$ starting from $T_1 = \Delta$.

(c) (*1 point*) Explain how the continuously compounded yield should behave (i.e., increasing or decreasing) in the following time steps if the continuously compounded forward rate is above the continuously compounded yield at the previous time steps and vice versa.

The response for this part is required on the paper provided to you.

The responses for all parts of this question are required on the paper provided to you.

7.

(*7 points*) For an interest rate derivative security V_t with maturity at time *T*, its price can be expressed as:

$$V_t = Z(t,T)E^*[g_T]$$

where $Z(t,T)$ is the price of a zero-coupon bond maturing at time *T*, g_T is the payoff at maturity, and $E^*[\cdot]$ is taken with respect to the *T*-forward risk-neutral measure.

Let $f(t,T_1,T_2)$ be the forward rate from T_1 to T_2 at initiation time t.

(a) (*1 point*) Show that $f(t,T_1,T_2) = E^*[r(T_1,T_2)]$, where $r(T_1,T_2)$ is the spot rate from T_1 to T_2.

Assume that under the *T*-forward risk-neutral measure, the diffusion process of $f(t,T_1,T_2)$ can be expressed as $\frac{df(t,T_1,T_2)}{f(t,T_1,T_2)} = \mu dt + \sigma dW_t$, where μ and σ are constants, and W_t is a Wiener process.

(b) (*1 point*) Determine the distribution of $\log(r(T_1,T_2))$ including the defining parameters.

Assume that $r(T_1,T_2)$ follows the Libor Market Model (LMM), and the price of a caplet with maturity T_2 and strike r_K is given by Black's formula:

$$Z(t,T_2)\left(f(t,T_1,T_2)N(d_1) - r_K N(d_2)\right),$$

where $d_1 = \frac{1}{\sigma\sqrt{T_2}} ln\left(\frac{f(t,T_1,T_2)}{r_K}\right) + \frac{1}{2}\sigma\sqrt{T_2}$ and $d_2 = d_1 - \sigma\sqrt{T_2}$.

7. **Continued**

You are given the following information about three caplets:

Maturity	Payoff	Strike r_K	Price	Current forward rates
				$f(0,0,0.25) = 1.0\%$
$T_2 = 0.5$	$1000 \times \max(r(0.25,0.5) - r_K, 0)$	1.1%	0.648	$f(0,0.25,0.5) = 1.1\%$
$T_3 = 0.75$	$1000 \times \max(r(0.5,0.75) - r_K, 0)$	1.2%	0.985	$f(0,0.5,0.75) = 1.2\%$
$T_4 = 1$	$1000 \times \max(r(0.75,1) - r_K, 0)$	1.3%	1.532	$f(0,0.75,1) = 1.3\%$

(c) (*1 point*) Calculate the forward volatility, σ_f^{i+1}, implied by the given caplet prices, where $i + 1$ refers to the caplet maturing at T_{i+1}.

Given a call option maturing at $T = 0.5$ with strike price K, on a zero-coupon bond with notional value of 1000 and maturity at $T = 1$, and the following information:

- The diffusion process of $f(t, T_1, T_2)$ is not martingale under T-forward measures where $T \neq T_2$.
- Assume that the volatility of forward rates depends only on time to maturity, $\sigma_f^{i+1} = S(T_{i+1})$ for some function $S(\cdot)$.
- The function $S(\cdot)$ is constant on each expiry period of the cap, i.e. (T_1=0.25, T_2=0.5), (T_2, T_3), (T_3, T_4).
- Assume the forward rates for any two maturities are perfectly correlated.

Under the T-forward risk-neutral measure, where $T = 0.5$, the following information is provided:

Current t	Forward rate	Random simulation from a standard normal
0.25	$f(0.25,0.5,0.75)$ $f(0.25,0.75,1)$	$z_1 = 0.1$ $z_2 = -0.2$

(d) (*3 points*)

(i) Simulate one outcome for each of the two forward rates in a single simulation step.

(ii) Describe how the call option on the zero-coupon bond can be priced using Monte Carlo simulations.

(e) (*1 point*) Describe two limitations of calibrating interest rate volatility using Black's formula and two potential alternatives to the LMM to address them.

8.

(*7 points*) Suppose $\gamma^* = 0.25, \sigma = 0.02$ and $f(0,t) = 0.05 + 0.02e^{-0.2t}$ is the instantaneous forward rate under the Hull-White Model:

$$dr_t = (\theta_t - \gamma^* r_t)dt + \sigma dX_t,$$

The zero-coupon bond price *Z(t,T)*, is given by

$$Z(t,T) = e^{A(t,T) - B(t,T)r_t},$$

where $B(t,T) = \frac{1}{\gamma^*}\left(1 - e^{-\gamma^*(T-t)}\right)$ and

$A(t,T)$ is a function of $B(t,T)$.

(a) (*2 points*) Calculate the price of a 3-month European call option on a zero-coupon bond that matures in 10 years with a face value of \$100 and a strike price of \$55.

The response for this part is to be provided in the Excel spreadsheet.

(b) (*1.5 points*) Derive

(i) the stochastic differential equation for the forward rate $f(t,T)$ from the Hull-White Model.

The response for this part is required on the paper provided to you.

(ii) the volatility of the forward rate.

The response for this part is required on the paper provided to you.

8. **Continued**

Consider the Cox-Ingersoll-Ross (CIR) model:

$$dr_t = 0.1(0.03 - r_t)dt + \sqrt{0.004 \cdot r_t}dX_t$$
and $r_0 = 0.023$.

(c) (*2 points*)

(i) Verify the technical condition that guarantees the interest rate is always positive is satisfied.

The response for this part is to be provided in the Excel spreadsheet.

(ii) Calculate the bond price $Z(0,10)$.

The response for this part is to be provided in the Excel spreadsheet.

You are considering alternative reference rates for interest rate derivative securities.

(d) (*1.5 points*)

(i) Describe ideal reference rate features.

The response for this part is required on the paper provided to you.

(ii) Explain why LIBOR fails to meet ideal reference rate features.

The response for this part is required on the paper provided to you.

(iii) Explain the attributes that the new reference rates would incorporate.

The response for this part is required on the paper provided to you.

The responses for all parts of this question are required on the paper provided to you.

9

(*9 points*)

(a) (*1.5 points*) Describe

(i) Forward Risk Neutral Pricing methodology.

(ii) The LIBOR market model (LMM) and its inputs and methodologies for pricing caps, floors and complicated securities.

You are asked to price a caplet struck at time T_i =1.5 and maturity T_{i+1} =2 under the LMM.

You are given the following information:

- Principal = \$100,000.
- The LIBOR spot rate= 2% for all maturities.
- The forward implied volatility σ_i=30% and is assumed to be constant over the period considered.
- Cap rate, r_K =3%.

(b) (*2 points*) Compute the value of the above caplet at time 0 under the LMM.

Let

- $Z(t, T)$ be the value of zero-coupon bond at time *t* with maturity T.
- $r_n(\tau, T)$ be the *n*-times compounded LIBOR at τ with maturity T, where $n = 1/\Delta$ and $\tau = T - \Delta$
- $f_n\ (0, \tau\ , T)$ be the *n*-times compounded forward rate at 0 for an investment at τ and maturity T.
- σ_f be the volatility of the forward rate implied from caplet prices.

In the LMM, the LIBOR rate $r_n(\tau, T)$ under the natural numeraire $Z(0, T)$ has a log-normal distribution with mean of $f_n\ (0, \tau\ , T)$ and variance of $\sigma_f^2\ \tau$.

$E_f^{*\tau}[r_n(\tau, T)]$ is the expected value of $r_n(\tau, T)$ with respect to the τ- forward dynamics.

9. Continued

(c) (*1 point*) Express the convexity adjustment term in terms of $f_n\ (0, \tau\ , T)$, σ_f, Δ and τ.

In a LIBOR-in-arrears swap, the following information is given:

- The principal is \$100 million.
- A fixed rate of 3% is received annually and LIBOR is paid.
- Payments are exchanged at the ends of years 1, 2, 3, 4.
- The yield curve is flat at 3% per annum (measured with annual compounding).
- All caplet volatilities are 20%.

The forward rate $f_n\ (0, \tau\ , T)$ under the regular swap is 3%.

(d) (*1.5 points*) Calculate the value of the LIBOR-in-arrears swap by using part (c).

Suppose that the forward rates $F_i(t)$ satisfy the stochastic differential equations below of the LMM under the measure P_{T_2} where the numeraire is the zero-coupon bond $Z(t, T_2)$ maturing at time T_2.

For $i = 1, 2$

$$dF_i(t) = \mu_i(t)F_i(t)dt + \sigma_i(t)F_i(t)dW_i(t)$$

where $W_1(t)\ and\ W_2(t)$ are correlated Brownian Motions under the measure P_{T_2} such that $dW_1(t)dW_2(t) = \rho dt$

$F_i(t)$ follows a log-normal diffusion process under the forward measure P_{T_i}.

In the swap market model (SMM) the forward swap rate $S_{0,2}(t)$ follows log-normal dynamics.

$$A_{0,2}(t) = \sum_{j=1}^{2} \tau_j\, Z\big(t, T_j\big)$$

$$S_{0,2}(t) = \frac{[Z(t,T_0) - Z(t,T_2)]}{\sum_{j=1}^{2} \tau_j Z(t,T_j)} = \frac{\left[1 - \prod_{i=1}^{2}\big(1+\tau_i F_i(t)\big)^{-1}\right]}{\sum_{j=1}^{2} \tau_j \prod_{i=1}^{j}\big(1+\tau_i F_i(t)\big)^{-1}}$$ is a martingale under the

swap measure P_A where $dS_{0,2}(t) = \sigma_{0,2}(t)S_{0,2}(t)dW(t)$,

$\sigma_{0,2}(t)$ is a deterministic time-dependent volatility and $W(t)$ is a Brownian Motion under the swap measure P_A.

9. **Continued**

Switching from the measure P_{T_2} to the swap measure P_A,

$$dF_i(t) = \mu_i^A(t)F_i(t)dt + \sigma_i(t)F_i(t)dW_i(t)$$

where $W_1(t)\ and\ W_2(t)$ are correlated Brownian Motions under the measure P_A such that $dW_1(t)dW_2(t) = \rho dt$.

(e) (*3 points*)

(i) Show by using Ito's lemma that

$$dS_{0,2}(t) = \sum_{i=1}^{2} \frac{\partial S_{0,2}(t)}{\partial F_i} \sigma_i(t)F_i(t)dW_i(t)$$

Hint: Consider the diffusion term of $dS_{0,2}(t)$ and use the following differential equations:

$$dF_i(t) = \mu_i^A(t)F_i(t)dt + \sigma_i(t)F_i(t)dW_i(t);\ i = 1, 2$$

(ii) Show by using part (i) and $dS_{0,2}(t) = \sigma_{0,2}(t)S_{0,2}(t)dW(t)$ that

$$\sigma_{0,2}(t)^2 = \frac{\sum_{i,j=1}^{2} \frac{\partial S_{0,2}(t)}{\partial F_i} \frac{\partial S_{0,2}(t)}{\partial F_j} \sigma_i(t)\sigma_j(t)F_i(t)F_j(t)\rho_{i,j}}{S_{0,2}(t)^2}$$

where $\rho_{1,2} = \rho_{2,1} = \rho$ and $\rho_{1,1} = \rho_{2,2} = 1$

(iii) Assess, by considering the distribution of $\sigma_{0,2}(t)$ in part (ii) whether the LMM and the SMM are compatible.

10.

(*6 points*) You want to hedge a long volatility swap on the ABC Index, *S*, with $1 million notional amount and a strike of 17%. There is one-year to expiration.

(a) (*0.5 points*) Determine the notional amount of the variance swap necessary to hedge the existing volatility swap. Assume you can sell variance swaps on the ABC Index at the strike of 17%.

The response for this part is required on the paper provided to you.

(b) (*1 point*) Determine the payoff on the hedged position if the realized volatility is 24%.

The response for this part is required on the paper provided to you.

You buy a European option at an implied volatility of σ and hedge to option expiry, *T*, at a constant hedge volatility σ_h.

σ_R is the realized volatility and Γ_h is the gamma of the hedging portfolio.

V_h is the option price assuming hedge volatility and V_i is the option price assuming implied volatility.

(c) (*2.5 points*) Show the present value of P&L at time t_0 is given by

$$PV[P\&L(i,h)] = V_h - V_i + \frac{1}{2}\int_{t_0}^{T} e^{\{-r(t-t_0)\}}\ \Gamma_h S^2(\sigma_R^2 - \sigma_h^2)dt.$$

The response for this part is required on the paper provided to you.

The ABC Index is currently trading at 4,000. Assume the implied and the realized volatility are both 20% and the ABC Index follows Geometric Brownian Motion and risk-free interest rates and dividends are zero. Consider a three-month at-the-money European call option. Assume 252 business days per year.

(d) (*1 point*) Calculate the standard deviation of the hedging error when rebalancing daily.

The response for this part is to be provided in the Excel spreadsheet.

(e) (*1 point*) Calculate the price of the call option assuming transaction costs of 1 basis point and daily rebalancing.

The response for this part is to be provided in the Excel spreadsheet.

11.

(*5 points*) Your analyst fits a curve described below to the volatility smile of a European put option on a non-dividend paying stock:

$$\sigma(S, K) = 0.3 * (S/K)^{1.6}, \text{ where } S = \text{stock price}; K = \text{strike price}$$

You are given:

- The curve is fitted based on the market prices of options with strike prices of \$60, \$70, \$80, \$90, and \$100, respectively.
- There are no other strike prices traded in the market
- The time-to-maturity = 1 year for all options
- The current stock price = \$100
- The risk-free interest rate = 0%

(a) (*3 points*)

(i) (*1 point*) Identify an arbitrage opportunity that is implied by your analyst's curve.

The response for this part is to be provided in the Excel spreadsheet.

(ii) (*2 points*) Show how you can take advantage of the arbitrage opportunity implied by your analyst's curve.

The response for this part is required on the paper provided to you.

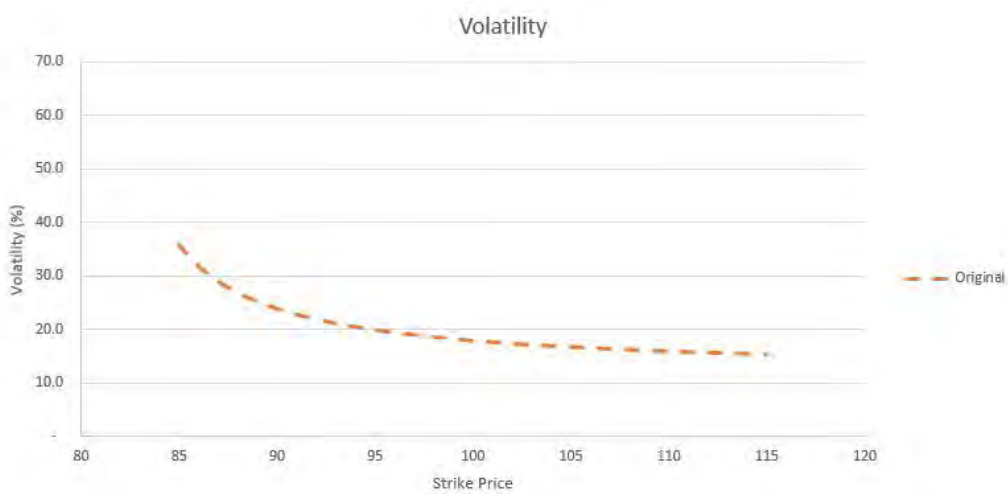
You believe that your analyst's curve correctly captured the dynamics of the implied volatility when the stock price is \$100. You decide to use a delta-hedging strategy to replicate the payoff of 1,000 identical put options, each with strike price = \$100 and time-to-maturity = 1 year.

(b) (*2 points*) Calculate the initial number of shares you need to trade. Clarify your trade as "buy" or "sell".

The response for this part is required on the paper provided to you.

12.

(*6 points*) You and your colleague are studying volatility hedging. You are given the following plot of the implied volatility of an equity index across different strike prices for options with 6-month maturities.



Strike K	90	100	110
Volatility σ	24%	18%	16%

(a) (*1 point*)

(i) Identify the common name of the above observation.

The response for this part is required on the paper provided to you.

(ii) Explain one reason for this observation.

The response for this part is required on the paper provided to you.

Your colleague made the following comment:

> "If the volatility is constant, there is no need to volatility hedge. But since the volatility can change by strike price, it's also important to ensure that the total Vega exposure of the portfolio is zero."

(b) (*1 point*) Critique the comment of your colleague.

The response for this part is required on the paper provided to you.

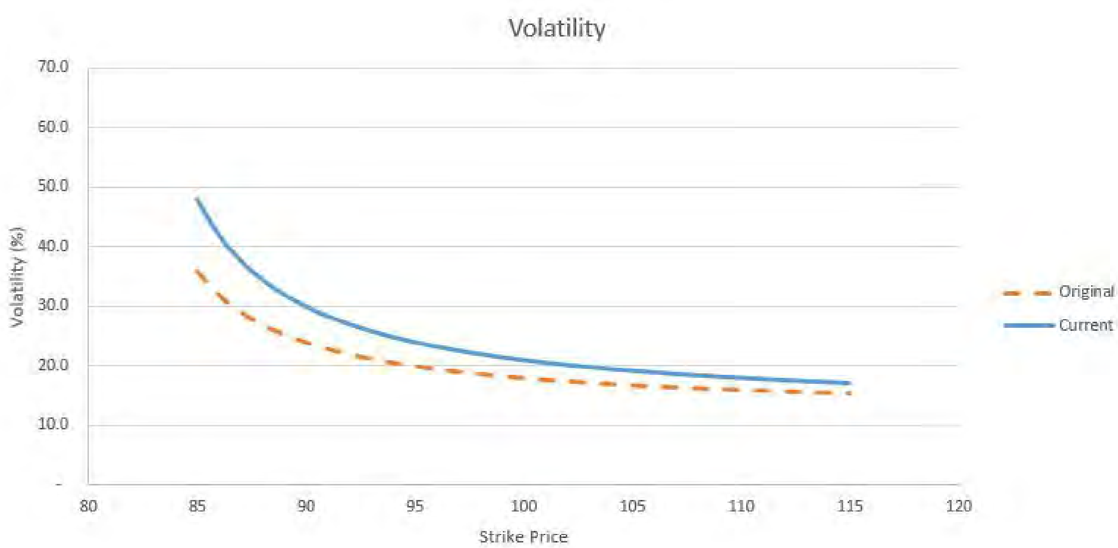
12. Continued

You learned several volatility hedging strategies: straddles and strangles, risk reversals, and butterflies.

Use the following assumptions for the questions below:

- Current index value $S = 100$
- Risk-free rate (continuously compounded) $r = 2\%$
- Three strike levels of interest: $K = 90, 100, 110$ always assuming 100 is the mid-point when applicable.
- Time to maturity = 0.5 years

You believe the volatility-strike plot will move from the current level given by the solid line above to the original dash line.



Strike K	90	100	110
Original Volatility σ	24%	18%	16%
Current Volatility σ	30%	21%	18%

12. **Continued**

(c) (*2.5 points*)

(i) (*0.5 points*) Choose one strategy from the above that you think is the most appropriate. Explain why.

The response for this part is required on the paper provided to you.

(ii) (*1.5 points*) Plot the Vega as a function of the underlying price.

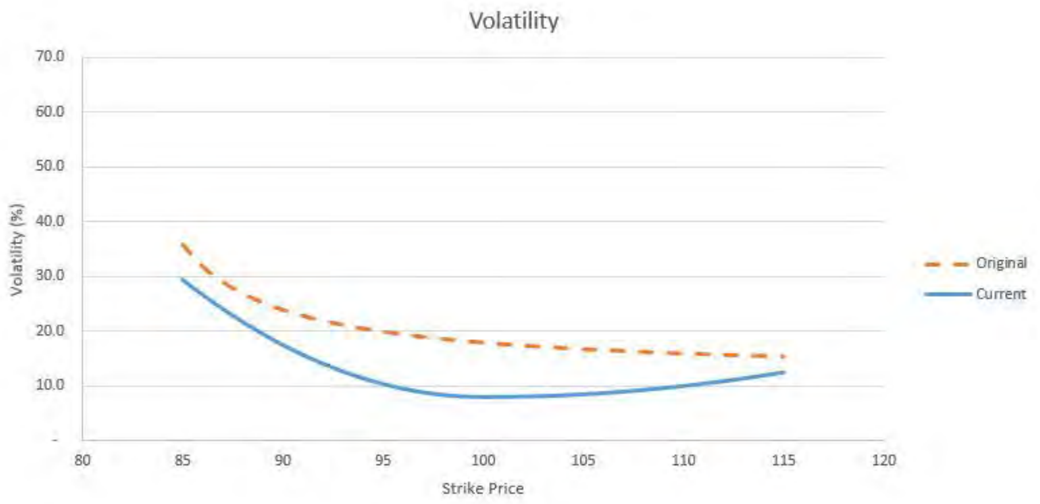
The response for this part is to be provided in the Excel spreadsheet.

(iii) (*0.5 points*) Calculate the maximum and minimum in the plot.

The response for this part is to be provided in the Excel spreadsheet.

12. Continued

You believe the volatility-strike plot will move from the current level given by the solid line below to the dash line.



Strikes K	**90**	**100**	**110**
Original Volatility σ	24%	18%	16%
Current Volatility σ	18%	8%	10%

(d) (*1.5 points*)

(i) (*0.5 points*) Choose one strategy from the above that you think is the most appropriate. Explain why.

The response for this part is required on the paper provided to you.

(ii) (*0.5 points*) Describe the construction of the strategy. Specify the long/short position of the strategies involved.

The response for this part is required on the paper provided to you.

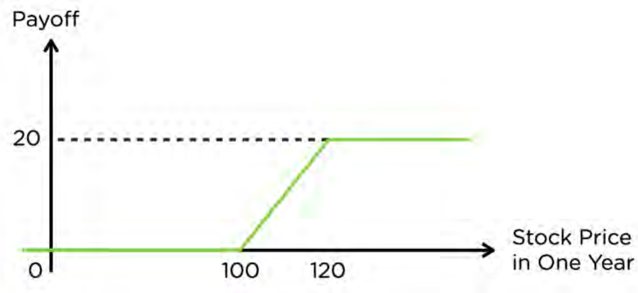
(iii) (*0.5 points*) Explain why the strategy is not Vega-neutral and describe, without calculation, how to construct it to be Vega-neutral

The response for this part is required on the paper provided to you.

The responses for all parts of this question are required on the paper provided to you.

13.

(*5 points*) Consider a one-year contingent claim on Stock XYZ, a non-dividend-paying stock. The payoff of the contingent claim in one year is as follows:



(a) (*0.5 points*) Construct a strategy to replicate the payoff of the contingent claim with only European options on Stock XYZ.

Assume that:

- The Black-Scholes framework applies.
- The continuously compounded risk-free interest rate is 0%.
- The price of Stock XYZ is 110.
- The implied volatility used for both valuation and hedging is 30%.

(b) (*0.5 points*) Compare and contrast realized volatility and implied volatility.

(c) (*1.5 points*)

(i) Calculate the Delta of this contingent claim.

(ii) Explain why the Delta is positive.

Regarding a long position in the contingent claims, your colleague made the following comments:

- Comment 1: As the price of the underlying stock moves away from the price range within the two strike prices, we expect the Delta of the contingent claim to converge to zero.
- Comment 2: The net Gamma exposure of the contingent claim is always positive.

(d) (*1 point*) Assess each of your colleague's comments above.

13. **Continued**

Your firm enters into a long position in 100 of contingent claims described above and immediately delta hedges the position using the underlying shares. You are given:

- Initially, the price of the stock is 110, the price of the 100-strike European call is 18.14, and the price of the 120-strike European call is 9.28.
- One day later, the price of the stock is 130, the price of the 100-strike European call is 33.56, and the price of the 120-strike European call is 20.40.

Assume your firm has not rebalanced the hedge and only whole shares can be bought or sold.

(e) (*1.5 points*)

(i) Calculate the profit or loss at the end of the next day from Delta hedging.

(ii) Explain why the profit or loss is not zero from Delta hedging.

The responses for all parts of this question are required on the paper provided to you.

14.

(*9 points*) The Generico VA company (GVA) is a new company looking to enter the Variable Annuity (VA) market and is considering various designs for its first product.

GVA is comparing two GMAB designs with short-term guarantees using the Black-Scholes model. Design 1 has a maturity at the end of 2 years with a minimum guaranteed compounding return of 3.5% per year for the 2-year period. Design 2 also has a 2-year duration, but with annual resets and no guaranteed return.

Assuming early redemptions are not allowed, a continuous annual risk-free rate of 3%, market volatility of 15%,

(a) (*1.5 points*) Calculate the risk-neutral probability that design 2 results in a larger pay off than design 1.

The CFO of GVA is interested in analyzing other guarantee base formulas for the proposed GMAB product. He has asked an analyst to put together a table of the last 4 years of stock returns in order to compare the different guarantee base formulas:

		Year (t)				
		0	1	2	3	4
	S_t	100	95	80	105	125
Benefit base	Reset	100				
	Compounding Roll-up (k = 3%)	100				
	Annual High Step up	100				

(b) (*1.5 points*)

(i) (*1 points*) Calculate the guaranteed benefit bases (in the table above) for the designs under consideration.

(ii) (*0.5 points*) Compare the performance of the different guarantee bases during a prolonged equity down-turn.

14. Continued

As an alternative to the proposed GMAB products, GVA is also considering a GMWB product but is concerned about its limited experience data for policyholder behavior.

(c) (*1 point*)

(i) Describe two types of policyholder behavior that could impact this type of product.

(ii) Describe how GVA could design their product to mitigate the risk of these behaviors in part (c)(i).

GVA's chief risk officer (CRO) is concerned about the potential losses of the GMWB product due to the exposures to market and interest rate risks. She recommended to hedge the product by using a combination of the underlying stock, 1-year risk-free zero-coupon bonds and a bank account-type asset. Below are the projected stock prices and sensitivities of the VA liability (assume continuous annual risk-free rate is 3%):

Time (t)	$\partial L/\partial S$	S_t	$\partial L/\partial r$
0	0.5	100	-.02

(d) (*1.5 points*) Construct a self-financing hedge portfolio, at time t=0, for the GMWB product based on the CRO's suggestion, and the parameters (in table above) determined by the company's hedging area.

Under the best-estimate economic scenario, the distributed earnings are in the table below:

Projection year	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Distributable Earnings (End of Year)	-1200	200	300	400	500	600	700

(e) (*2 points*)

(i) (*1 point*) Calculate the net present value of the distributable earnings and the profit margin using a hurdle rate of 8%.

(ii) (*1 point*) Critique the profit-testing approach used, in part (e)(i), to evaluate the profitability of the VA product.

14. Continued

(f) (*1.5 points*)

(i) (*0.5 points*) Contrast the similarity and difference between actuarial profit-testing and no-arbitrage pricing practices.

(ii) (*1 point*) Explain the reasons that no-arbitrage pricing is not directly used in the insurance industry.

15.

(*9 points*) QFI Life incurred severe losses in the recent market crash with its variable annuity (VA) business. To reduce its market risk exposure, QFI Life has been exploring a new structured product based variable annuity (spVA).

Your co-worker David proposes a spVA product with buffered and capped structures, where the buffer rate = B% and the cap rate = C%.

(a) (*1.5 points*) Assume an initial deposit of \$100 is made to this spVA product when the reference asset value is \$100.

(i) Express the buffer level and the cap level, in terms of B% and C%.

The response for this part is required on the paper provided to you.

(ii) Sketch the maturity payoff of this spVA against the reference asset value (label the buffer and cap levels).

The response for this part is required on the paper provided to you.

(iii) Specify and justify a portfolio of bonds and options that replicates the maturity payoff of the spVA.

The response for this part is required on the paper provided to you.

The embedded options of the spVA product with buffered and capped structure are modeled using the following Black-Scholes parameters:

spVA with buffered and capped structures	
Underlying Asset (S_0)	100
Dividend Yield (q)	0%
Implied Volatility (σ)	10%
Term (t)	1 year
Risk-Free Rate (r)	2%
Buffer Rate% (B)	10%

(b) (*2 points*) Calculate and determine whether the fair cap level is 5.5%, 7.5% or 10.5% for this spVA product to address the market risk.

The response for this part is to be provided in the Excel spreadsheet.

15. **Continued**

QFI Life has $100M of inforce VA GMMB business that is currently at-the-money and with remaining time-to-maturity of 1 year. The market risk of the VA business is modeled as a put option with the parameters in the table below.

VA (Inforce Business)	
Underlying Asset – Price at Issue	100
Underlying Asset – Current Price (S_0)	100
Dividend Yield (q)	0%
Implied Volatility (σ)	10%
Term (t)	1 year
Continuous Risk-Free Rate (r)	2%
Maturity Benefit	Max (Fund Value, 90% Deposit)

QFI Life expects sales of $50M of new spVA described in part (b)

(c) (*1 point*) List four key metrics for evaluating risk management strategy, together with their objectives.

The response for this part is required on the paper provided to you.

(d) (*3 points*)

(i) Explain how the spVA sales could be used to mitigate the market risk of inforce business, in terms of the embedded option and pricing control through the cap.

The response for this part is required on the paper provided to you.

(ii) Calculate the total Vega for QFI Life's business, including the $50M of new spVA sales .

The response for this part is to be provided in the Excel spreadsheet.

(iii) Calculate the change in Vega, assuming implied volatility suddenly increases to 20%.

The response for this part is to be provided in the Excel spreadsheet.

(iv) Assess the impact of the spVA business on the change in Vega, based on results in part (ii) and part (iii).

The response for this part is required on the paper provided to you.

15. Continued

You learned that target volatility fund can be a useful tool to manage market risk exposure, and would like to assess the benefit, assuming all deposits are made to the target volatility fund.

Target Volatility Fund		
Target Volatility	10%	
Maximum Allocation	110%	
Scenario	**Scenario 1**	**Scenario 2**
Underlying Asset (S_0)	100	100
Dividend Yield (q)	0%	0%
Realized Volatility (σ_R)	10%	20%
Implied Volatility (σ)	10%	20%
Risk-Free Rate (r)	2%	2%

(e) (*1.5 points*) Under Scenario 1 and Scenario 2, respectively

(i) Determine the allocation to the underlying asset.

The response for this part is required on the paper provided to you.

(ii) Calculate the total Vega of QFI Life's VA business (i.e., no spVA).

The response for this part is required on the paper provided to you.

(iii) Explain how the target volatility fund reduces the market risk of inforce business, based on results in part (i) and part (ii).

The response for this part is required on the paper provided to you.

****END OF EXAMINATION****