QFI Quantitative Finance Exam

Exam QFIQF

Date: Wednesday, October 27, 2021

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 15 questions numbered 1 through 15 with a total of 100 points.

   The points for each question are indicated at the beginning of the question.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

3. Each question part or subpart should be answered either in the Excel document or the paper provided as directed. Graders will only look at the work as indicated.

4. In the Excel document, answers should be entered in the box marked ANSWER. The box will expand as lines of text are added. There is no need to use special characters or subscripts (though they may be used). For example, β_1 can be typed as beta_1 (and ^ used to indicate a superscript).

5. Prior to uploading your Excel files, each file should be saved and renamed with your five-digit candidate number in the filename.

6. The Excel file that contain your answers must be uploaded before the five-minute upload period expires.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, hand in all your written-answer sheets to the Prometric center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

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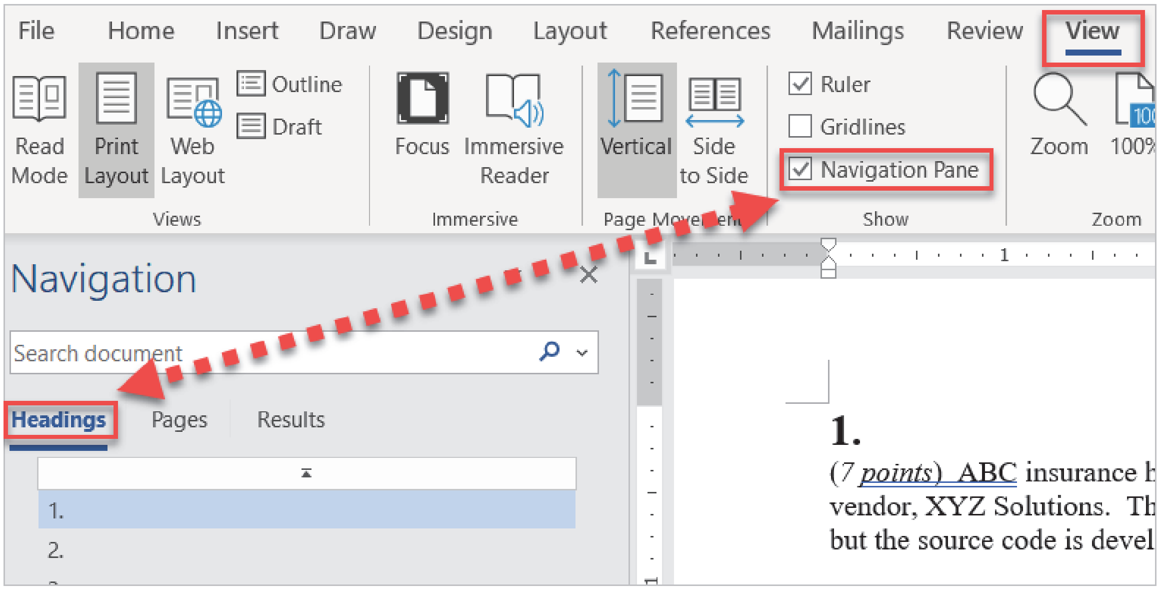
Navigation Instructions

Open the Navigation Pane to jump to questions.

Press Ctrl+F, or click View > Navigation Pane:



The responses for all parts of this question are required on the paper provided to you.

1.

(*6 points*) Let W_t be a standard Wiener process defined on the interval $[0, T]$. For $t \in [0, T]$, let X_t be defined as

$$X_t = \int_0^t W_u du.$$

(a) (*1 point*) Explain why X_t is a normally distributed random variable for $t > 0$.

(b) (*3 points*) Compute

(i) $E[X_t]$.

(ii) $Var[X_t]$.

Let Y_t be defined as

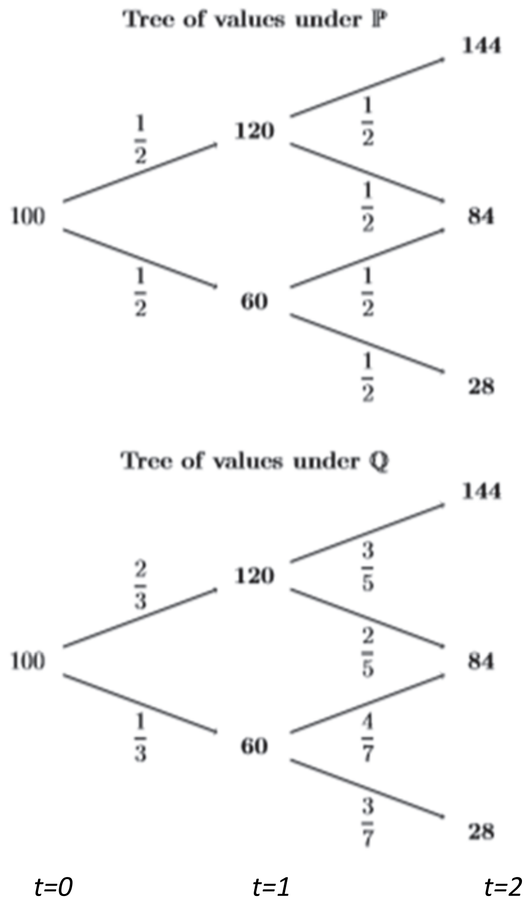
$$Y_t = \int_0^t \sqrt{|W_u|} dW_u.$$

(c) (*2 points*) Compute $Var[Y_t]$.

The responses for all parts of this question are required on the paper provided to you.

2.

(*6 points*) Consider a price process A_t that evolves in discrete, integer time t according to a recombining binomial tree. The set of all possible values of the process are shown below for $t = \{0, 1, 2\}$ under two different probability measures $\mathbb{P}$ and $\mathbb{Q}$. Assume the risk-free rate is 0%.



(a) (*0.5 points*) Show that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures on the probability space implied by the price process A_t.

(b) (*2 points*) Determine if the price process A_t is a:

(i) $\mathbb{Q}$-martingale.

(ii) $\mathbb{P}$-martingale.

2. **Continued**

(c) (*1 point*) Calculate the values of the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ for all paths through the tree, (i.e. up-up, up-down, down-up, down-down nodes).

(d) (*0.5 points*) Evaluate the process $\xi_t = E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_t\right)$ at time $t = 1$ for both up and down nodes where $\mathcal{F}_t$ is the filtration history up to time *t*.

Consider a claim *X* that pays a fixed amount of $20 at time *t* = 2 if the price process A_t attains a value of more than $75, otherwise it pays nothing.

(e) (*2 points*) Show numerically that $E^{\mathbb{Q}}[X] = E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X\right]$ at time 0 by using the results in part (d).

The responses for all parts of this question are required on the paper provided to you.

3.

(*6 points*) Consider one risky asset S_t and one risk-free asset B_t with risk-neutral price dynamics:

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$
$$dB_t = rB_t dt, B_0 = 1$$

where $W_t^{\mathbb{Q}}$ is a standard Wiener process under the risk-neutral measure $\mathbb{Q}$ and r, σ are constants.

Let X be a European claim with exercise time T, and price $\pi_t(X)$ at time $t < T$. Define $\pi_t^d(X) = B_t^{-1}\pi_t(X)$ to be its discounted price.

Suppose the claim can be replicated by a self-financing portfolio consisting of α_t units in the risky asset S_t and β_t units in the risk-free asset B_t, i.e.:

$$\pi_t(X) = \alpha_t S_t + \beta_t B_t,$$
$$d\pi_t(X) = \alpha_t dS_t + \beta_t dB_t.$$

(a) (*2 points*)

(i) Determine the stochastic differential equation satisfied by the discounted price process $S_t^d = B_t^{-1} S_t$.

(ii) Explain why $\pi_T^d(X) = \pi_t^d(X) + \int_t^T \alpha_u \sigma S_u^d dW_u^{\mathbb{Q}}$.

(iii) Show that the discounted derivative prices $\pi_t^d(X), t < T$ form a $\mathbb{Q}$-martingale using part (a) (ii).

3. **Continued**

Let $C_t(K,T)$ be the price of the call option at time *t* with strike price *K* and maturity *T* and $P_t(K,T)$ be the price of the put option at time *t* with the same maturity and strike price.

(b) (*2 points*) Prove that $C_t(K,T) - P_t(K,T) = S_t - Ke^{-r(T-t)}, t < T$.

A European chooser option, *V*, with exercise time $T_c < T$ is the option to choose, at time T_c, between a put and a call with identical maturity *T* and strike price *K*. Its payoff at T_c is $\max\left(P_{T_c}, C_{T_c}\right)$.

(c) (*2 points*) Show that $\pi_t^d(V) = P_t(K,T) + C_t\left(Ke^{-r(T-T_c)}, T_c\right), t < T_c$.

The responses for all parts of this question are required on the paper provided to you.

4.

(*6 points*) Consider the probability space $\Omega = \{u, d\}^n$, i.e. the set of all possible n-tuples of the letters u and d, and define a discrete random process $X = \{X_k\}_{1 \le k \le n}$ given by

$$X_k(\omega) = \sum_{i=1}^{k} \left[1_u(\omega_i) - 1_d(\omega_i) \right],$$

with

$$\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \Omega$$

and $1_u, 1_d$ the indicator functions of u, d, respectively.

Using the same notation as above, let $\mathcal{F}_k$, $1 \le k \le n$, be the σ-algebra generated by the k outcomes below:

	Position (1, 2, 3, …., *k-1*, *k*, …, *n*)	Value of ω_k
Outcome 1	$(u, d, d, \dots, d, d, \dots, d)$	d
Outcome 2	$(u, u, d, \dots, d, d, \dots, d)$	d
…		
Outcome (*k-1*)	$(u, u, u, \dots, u, d, \dots, d)$	d
Outcome *k*	$(u, u, u, \dots, u, u, \dots, d)$	u

In other words, outcome *l* has *u*'s in first *l* positions and *d*'s in the remaining *(n-l)* positions for $l \le k$.

Finally, define a probability measure $\mathbb{P}$ on any set $A \in \mathcal{F}_k$ by

$$\mathbb{P}(A) = \sum_{\omega \in A} \left(\prod_{i=1}^{n} p(\omega_i) \right), \text{ with } p(u) = p \text{ and } p(d) = 1 - p$$

where $0 < p < 1$.

(a) (*1 point*) Determine whether X is adapted to the filtration $\{\mathcal{F}_k\}_{1 \le k \le n}$.

4. **Continued**

(b) (*2 points*) Verify that $E^{\mathbb{P}}\left(E^{\mathbb{P}}\left(X_2 \middle| \mathcal{F}_2\right)\right) = E^{\mathbb{P}}\left(X_2\right)$ by direct computation.

For $1 \le k \le n$, let $I_k = \sigma\left(X_k\right)$ be the smallest σ-algebra that makes X_k I_k-measurable. Define $I = \left\{I_k\right\}_{1 \le k \le n}$.

Now consider a non-dividend paying stock with price S_k at time k, given by

$$S_k = S_0\left(\frac{1}{p} - 1\right)^{X_k}, 1 \le k \le n, 0 < p < 1, \text{and } 0 < S_0 < \infty.$$

(c) (*1.5 points*) Show that $S_k, 1 \le k \le n,$ forms a martingale with respect to $\left(\Omega, I, \mathbb{P}\right)$.

(d) (*1.5 points*) Find the value of p for which $E^{\mathbb{P}}\left(\sqrt{S_3} \middle| I_2\right) = \sqrt{S_2}$.

5.

(*6 points*)

(a) (*1 point*) Describe forward rate agreements, forward contracts, and interest rate swaps.

The response for this part is required on the paper provided to you.

All rates provided below are continuous compounding, swaps have semi-annual coupon payment.

t	T_0 (spot rate time 0)	$T_{0.5}$ (spot rate time 0.5)
0.5	2.00%	1.00%
1	2.50%	1.50%
1.5	3.00%	2.00%
2	3.50%	2.50%
2.5	3.50%	2.50%
3	4.00%	3.00%

ABC bank quotes 96.08 for a one-year forward contract on 1-year zero-coupon bond.

(b) (*1.5 points*) Determine arbitrage strategy based on the above data.

The response for this part is required on the paper provided to you.

(c) (*1 point*) Calculate the 2-year swap rate and the value of the swap at time 0.

The response for this part is to be provided in the Excel spreadsheet.

(d) (*1 point*) Calculate the value of the 2-year swap in part (c) at time 0.5 (after cash payment).

The response for this part is to be provided in the Excel spreadsheet.

(e) (*1.5 points*) Calculate the forward swap rate of the 2-year forward swap contract with expiry of 1 year.

The response for this part is to be provided in the Excel spreadsheet.

6.

(*6 points*) You are an actuarial analyst at the hedging department. The company is concerned that interest rates will decline in the next 6 months, at which time it will have to reinvest treasury notes worth $100 million maturing in 6 months. The company is first considering hedging the interest rate decline with a forward rate agreement (FRA).

Assume that the forward yield curve is a step function and the rates are continuously compounded:

$$f_t = \begin{cases} f_0 & 0 \le t < 1 \\ 2.99\% & 1 \le t < 2 \\ 3.33\% & 2 \le t < 5 \\ 4.06\% & 5 \le t < 10 \end{cases}$$

The current market par swap rates for swaps with annual payments are as follows:

Maturity (years)	Fixed Rate (%)
1	2.0
2	s_2
5	s_5
10	3.5

(a) (*1 point*) Define the following terms:

(i) forward rate f (0,2,5)

(ii) par swap rate s_2

The response for this part is required on the paper provided to you.

(b) (*2 points*) Calculate the corresponding rates f_0, s_2, s_5 such that the forward curve matches the swap curve.

The response for this part is to be provided in the Excel spreadsheet.

6. Continued

Instead of hedging with an FRA, the company could also hedge the interest rate risk by entering into the long position of Eurodollar futures contracts where the underlying instrument is 3-month LIBOR.

(c) (*1.5 point*) Recommend which hedging instrument to use, and justify your recommendation with 2 supporting arguments.

The response for this part is required on the paper provided to you.

Your company would like to explore using option strategies on the underlying 6-month Treasury bill as protection against the decline in interest rates. Strike price K is equivalent to the forward rate at time zero. Your company wants to limit the cost of this protection.

(d) (*1.5 points*) Construct an appropriate option strategy for each of the two objectives below:

(i) Hedge against the decline in interest rates but still have some upward potential profit from an increase in interest rate.

(ii) Hedge against a significant decline in interest rates such that it will retain some risk but spend less on the protection.

The response for this part is required on the paper provided to you.

The responses for all parts of this question are required on the paper provided to you.

7.

(*7 points*) Consider a world with two time periods (t_0, t_1) and (t_1, t_2), and for any given time t, there are two possible states.

Assume that $t_k = t_0 + k$ for $k = 1, 2,$ and there are liquid markets for the following instruments:

- A default-free savings account.
- A forward rate agreement (FRA) contracted at time t_0 with payoff at time t_2 written on the LIBOR rate L_1, a borrowing rate from time t_1 to t_2, which is unknown at t_0,
- A 2-year default-free zero-coupon bond, with maturity value \$1 and current price B_0.

Let

- F_0 = *forward* rate contracted at time t_0 on L_1,
- r_0 = spot rate at t_0,
- r_1^i = spot rate at time t_1, $i = u, d$.

(a) (*1 point*) Write down the matrix equation implied by the Fundamental Theorem of Finance.

7. **Continued**

You were given the following:

- arbitrage-free price of a 1-year zero-coupon bond = 0.980392
- arbitrage-free price of a 2-year zero-coupon bond = 0.960742

For time t_2,

State	Risk-neutral probabilities	State price
{u,u}	0.42	
{u,d}	0.28	0.267814

(b) (*2 points*) Calculate the following:

(i) Spot rate r_0 at time t_0

(ii) Spot rates r_1^i at time t_1 for $i = u, d$

(c) (*2 points*) Show that F_0 is not an unbiased estimator of L_1 under classical risk-neutral measure.

(d) (*2 points*) Show that F_0 is an unbiased estimator of L_1 under the forward measure with numeraire equals to the two-period zero-coupon bond (Zero-coupon bond maturity at time $T = t_2$).

8.

(*9 points*) You were given the short rate model with the following stochastic differential equation:

$$dr_t = a\left(b - r_t\right)dt + \sigma dW_t$$

where *a, b, σ* are constant and W_t is a standard Wiener process with respect to the measure $\mathbb{P}$.

Let *B* = *B(r,t,T)* be the price at time *t* of the default-free discount bond with maturity *T*.

Consider a self-financing portfolio, with a long position of η units of a short-term bond with maturity date T_1 and a short position of θ units of long-term bond with maturity date $T_2 > T_1$ chosen to eliminate interest rate risk.

$$P = \eta B_1\left(t, T_1\right) - \theta B_2\left(t, T_2\right)$$
$$dP = \eta B_1\left(t, T_1\right) - \theta dB_2\left(t, T_2\right)$$

(a) (*2.5 points*) Show that

$$\frac{r_t B_1 - \dfrac{\partial B_1}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2 B_1}{\partial r^2}\sigma^2}{\dfrac{\partial B_1}{\partial r}} = \frac{r_t B_2 - \dfrac{\partial B_2}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2 B_2}{\partial r^2}\sigma^2}{\dfrac{\partial B_2}{\partial r}}$$

The response for this part is required on the paper provided to you.

8. Continued

Denote by $m(t)=\dfrac{r_t B-\dfrac{\partial B}{\partial t}-\dfrac{1}{2}\dfrac{\partial^2 B}{\partial r^2}\sigma^2}{\dfrac{\partial B}{\partial r}}$,which is independent of the maturity of the bond from the result in part (a), and define the market price of risk $\lambda(r_t,t)=\dfrac{a(b-r_t)-m(t)}{\sigma}$.

(b) (*0.5 points*) Show that the price of a default-free discount bond satisfies the following partial differential equation

$$\frac{\partial B}{\partial r}\left[a(b-r_t)-\sigma\lambda\right]+\frac{\partial B}{\partial t}+\frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2-r_t B=0$$

The response for this part is required on the paper provided to you.

(c) (*1 point*) Describe the key features of this interest rate model.

The response for this part is required on the paper provided to you.

You are given data on one month T-bill rates from 1960 to 2010.

(d) (*1 point*) Explain how to estimate the interest rate model parameters, using the given data. Identify the estimated parameters that can be used in pricing interest rate derivative.

The response for this part is required on the paper provided to you.

Given the following information:

- Long-run mean of the spot rate: b = 0.05
- Speed of mean reversion: a = 0.25
- Absolute interest rate volatility: σ = 0.015
- Market price of interest rate risk: λ = -0.1

(e) (*1.5 points*) Calculate the default-free discount bond price with 30-year maturity with r = 0.1%, 5%, and 10%, respectively.

The response for this part is to be provided in the Excel spreadsheet.

8. Continued

(f) (*1.5 points*) Generate the yield curves for the same set of spot rates in part (e) with different maturities,1 through 30 years.

The response for this part is to be provided in the Excel spreadsheet.

Since the financial crisis, negative interest rates have been used extensively. However, you noticed that the company's practice is still applying a zero floor to the scenarios generated from this interest rate model.

(g) (*1 point*) Analyze the impact of zero floor on

(i) cost of guarantees;

(ii) discount factors and best estimate values;

(iii) martingality of economic scenarios and company's capital position.

The response for this part is required on the paper provided to you.

The responses for all parts of this question are required on the paper provided to you.

9.

(*9 points*) You have been asked to formulate an affine term structure model within the HJM framework.

You are given the generalized affine term structure model for the short-term interest rate r_t.

$$dr_t = (\theta_t - \gamma_t r_t)\,dt + \sqrt{\sigma_t^2 + a_t r_t}\,dX_t$$

where $a_t, \gamma_t, \theta_t, \sigma_t$ are functions of t only, X_t is a standard Brownian motion and the term $\sigma_t^2 + a_r r_t$ has zero probability of becoming negative.

The price of a zero-coupon bond with \$1 principal at time t with maturity date T is given by $Z(r,t,T) = e^{A(t,T)-B(t,T)r}$.

Let $f(t,T) = -\dfrac{\partial \ln Z(t,T)}{\partial T}$ be the continuously compounded instantaneous forward rate.

Let $\sigma_f(t,T)$ be the volatility of the instantaneous forward rate.

(a) (*1 point*) Show by using Ito's lemma that

$$df(t,T) = \left[\frac{\partial B(t,T)}{\partial T}(\theta_t - \gamma_t r_t) + r_t \frac{\partial^2 B(t,T)}{\partial t \partial T} - \frac{\partial^2 A(t,T)}{\partial t \partial T}\right] dt + \frac{\partial B(t,T)}{\partial T}\sqrt{\sigma_t^2 + a_t r_t}\,dX_t.$$

(b) (*1 point*) Describe the necessary inputs for pricing derivative securities within the HJM framework.

9. Continued

Consider the following short rate models:

1) Ho-Lee model for the short-term interest rate r_t:

$$dr_t = \theta_t dt + \sigma dX_t$$

where θ_t is a function of t only, σ is a constant, and X_t is a standard Brownian motion.

2) Black and Karasinski model for the short-term interest rate r_t:

$$dr_t = r_t\left(\theta_t + \frac{\sigma_t^2}{2} - \gamma_t \ln r_t\right)dt + \sigma_t r_t dX_t$$

where γ_t is non-zero and X_t is a standard Brownian motion.

(c) (*1 point*) Assess whether the Ho-Lee model and the Black and Karasinski model belong to the class of generalized affine models. Justify your answer.

(d) (*2 points*) Demonstrate that under the Ho-Lee model:

(i) $\sigma_f(t,T) = \sigma$ using part (a).

(ii) $f(t,T) = r_t + f(0,T) - f(0,t) + \sigma^2 t(T-t)$ where r_t is the short rate.

(e) (*2 points*) Demonstrate that under the Ho-Lee model:

(i) $Var\left[r_{T_0}\right] = T_0\sigma^2$

(ii) $Var\left[\log\left(Z\left(r_{T_0}, T_0; T_B\right)\right)\right] = \sigma^2 \cdot T_0\left(T_B - T_0\right)^2$

where $Z\left(r_{T_0}, T_0; T_B\right)$ is the value at time T_0 of a zero-coupon bond with maturity date T_B with $T_B > T_0$.

9. Continued

You are asked to price a 2-year maturity at the money (ATM) European call option on a zero-coupon bond that will mature in 10 years under the Ho-Lee model at time 0.

Principal = 1. Exercise price $K = \frac{Z(0,10)}{Z(0,2)}$.

The initial term structure of interest rates is given by:

$f(0,T) = 0.02 + 0.002T$, $\sigma_0 =$ Initial forward rate volatility=0.005.

The risk-neutral dynamics of forward rates are fully characterized by the volatility of forward rates.

(f) (*2 points*)

(i) Calculate exercise price K.

(ii) Compute the value at time $t = 0$ of the above European call option on the zero-coupon bond.

The responses for all parts of this question are required on the paper provided to you.

10.

(*6 points*) A company XYZ issue a 3-year zero-coupon bond $Z(0,3)$ with face value 100 million and an option to return the bond in exchange of 95 million at time 1. Your colleague came up with a hedge plan to mitigate the interest rate risk of the new issued bond by using a 5-year zero-coupon bond $Z(0,5)$ with face value 100 million. For simplicity, he assumed that the hedge plan would be in effective on the same date as the issue date of the company bond.

The current interest rate is $r_0 = 2\%$, the Vasicek model parameters are $\sigma = 2.21\%, \gamma^* = 0.4653, \overline{r}^* = 6.34\%$ for the formula:

$$Z(r,t,T) = e^{A(t,T)-B(t,T)r},$$
$$B(t,T) = \frac{1}{\gamma^*}\left(1 - e^{-\gamma^*(T-t)}\right),$$
$$A(t,T) = \left(B(t,T) - (T-t)\right)\left(\overline{r}^* - \frac{\sigma^2}{2(\gamma^*)^2}\right) - \frac{\sigma^2 B(t,T)^2}{4\gamma^*}$$

Your colleague considered a Vasicek model of interest rates for the hedge. He has computed the following items in his calculation:

T	1	2	3	4	5
B(0,*T*)	0.80	1.30	1.62	1.81	1.94
A(0,*T*)	-0.01	-0.04	-0.09	-0.14	-0.19
B(1,*T*)	0.00	0.80	1.30	1.62	1.81
A(1,*T*)	0.00	-0.01	-0.04	-0.09	-0.14

(a) (*3 points*) Calculate the option price for the issued bond at $t = 0$.

He also plans to build a replicating portfolio for the company bond using the 3-year zero-coupon bond and cash.

(b) (*1 point*) Specify an appropriate replicating portfolio.

10. Continued

Your colleague wants to hedge the option V from part (a) on the 3-year bond by constructing a hedge portfolio Π. The Vasicek model gives the calculation result of $\frac{\partial V}{\partial r} = -66.3$ for the hedge portfolio:

$$\Pi(r,t) = -V(r,t) + 100 \cdot \Delta \cdot Z(0,3)$$
$$\Delta = \text{Number of zero-coupon bond } Z(0,3)$$

(c) (*2 points*) Compute the position of $Z(0,5)$ and cash position of the hedge portfolio when the underlying bond is replicated as in part (b).

11.

(*6 points*) Table 1 below shows the market data of a European call option on a non-dividend paying stock XYZ.

Table 1: Market data on XYZ stock price and its call option

Option maturity as of Day 1 = 3 years Option strike price = 80 Option size = 1 (i.e. One option contract has 1 underlying share) No changes in interest rate from Day 1 to Day 30						
Day	1	2	3	4	…	30
Stock price	80	70	75	82		80
Option price	12.25	12.25	12.22	12.30		12.07
Option delta	0.610	0.535	0.562	0.638		0.608

On Day 1, you sold 1000 of this option and immediately started delta-hedging. On Day 4, you liquidated all your options and shares of the XYZ stock. All your trades are conducted at the prices shown in Table 1. Transaction cost (including the borrowing cost, if any, that you incurred as part of your hedging strategy) should be ignored for this question.

(a) (*3 points*) Calculate your cumulative total profit or loss on Day 4 under the following circumstances, respectively:

(i) You rebalanced your hedge position daily.

(ii) You never rebalanced your hedge position.

The response for this part is to be provided in the Excel spreadsheet.

11. Continued

Your analyst compiled Table 2 below that shows his explanations for the observations on the option and stock prices in Table 1. However, he was unable to explain observation 4.

Table 2:

Observation of the option and stock price movement	Analyst's explanation
1. The option price did not change from Day 1 to Day 2, while the stock price decreased from Day 1 to Day 2	The sticky strike rule
2. The option price decreased from Day 2 to Day 3, while the stock price increased from Day 2 to Day 3	The sticky delta rule
3. Both the option price and the stock price increased from Day 3 to Day 4	The local volatility model
4. The option price on Day 30 is lower than on Day 1, while the stock price on Day 30 is same as on Day 1	

(b) (*2.5 points*) Determine whether each of the three explanations provided is valid or not. Explain why.

The response for this part is required on the paper provided to you.

(c) (*0.5 points*) Provide your explanation for observation 4.

The response for this part is required on the paper provided to you.

12.

(*9 points*) For an exotic option *E* with the following payoff:

- $m\,(S - 150) + 50$, if S > 150
- $2\,(S - 100) + 50$, if S < 100
- 50, otherwise,

where S is an underlying equity with no dividend; *m* is constant.

(a) (*1 point*)

(i) Sketch the payoff graph for option *E* using $m = 2$.

(ii) Build a static hedging strategy with vanilla options to hedge the equity risk.

The response for this part is required on the paper provided to you.

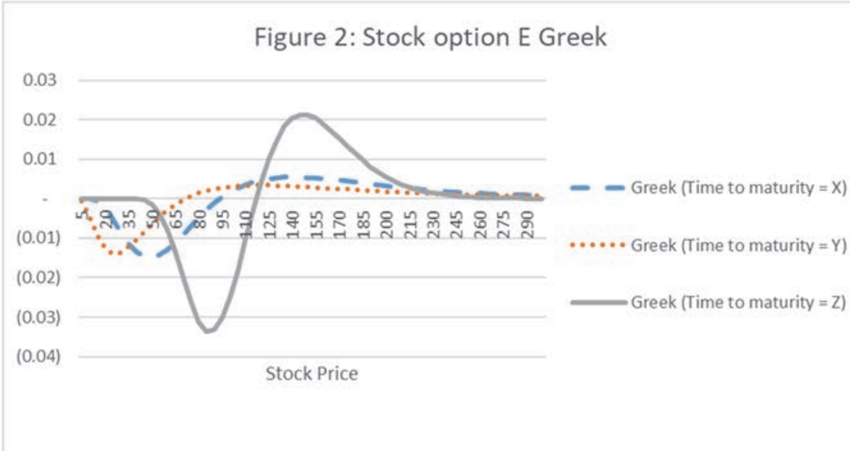
(b) (*3 points*)

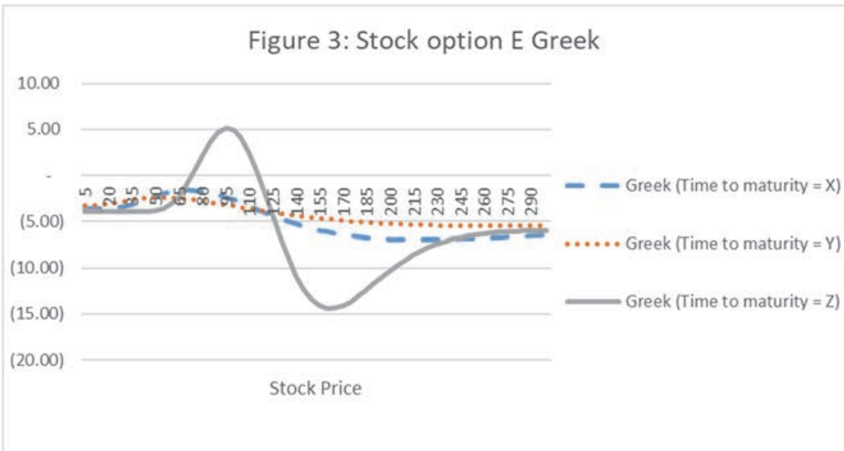
(i) Define the following Greeks: Delta, Gamma, Vega, and Theta.

(ii) Sketch Delta graph for option *E* using $m = 2$ and justify your answers. (Hint: Build from vanilla options.)

(iii) Determine which figure corresponds to Gamma, Vega, and Theta, respectively. Justify your answers.

12. Continued





The response for this part is required on the paper provided to you.

Your coworker proposes to trade on the convexity of volatility skew with option *E*.

(c) (*1.5 points*)

(i) Explain what the volatility skew is.

(ii) List three reasons why the volatility skew exists.

(iii) Explain why option *E* is not the suitable vehicle to trade on convexity of volatility skew.

The response for this part is required on the paper provided to you.

12. Continued

You are given the following for vanilla options on non-dividend paying stock.

Underlying S	100	100	100	100
Strike K	50	100	150	200
Volatility σ	27.7%	23.3%	21.2%	21.4%
T-t	5	5	5	5
r	2%	2%	2%	2%
d_1	1.59	0.45	-0.41	-1.00
d_2	0.97	-0.07	-0.88	-1.48
$N(d_1)$	0.94430	0.67450	0.34188	0.15909
$N(d_2)$	0.83487	0.47287	0.18905	0.06979
$N'(d_1)$	0.11236	0.36014	0.36718	0.24241
$N'(d_2)$	0.24836	0.39802	0.27053	0.13395

Your coworker is asking for your help to build a Vega-neutral option E^* to trade on the convexity of volatility skew, expecting

- Increase in the volatility convexity;
- Higher future volatility than the current implied volatility;
- Symmetric payoff against stock price, centered at the current level.

Option E^* = Option E – 2 Call (strike = 100) + m Put (strike = K^*)

(d) (*1.5 points*)

(i) Determine K^* for option E^*.

The response for this part is to be provided in the Excel spreadsheet.

(ii) Solve for m so that option E^* is Vega-neutral.

The response for this part is to be provided in the Excel spreadsheet.

12. Continued

The stock price decreases from 100 to 80 the next day, impacting the implied volatilities as below. (Time decay of 1 day is ignored.)

Underlying S	80	80	80	80
Strike K	50	100	150	200
Volatility σ	27.7%	30.0%	38.9%	63.1%
T-t*	5	5	5	5
r	2%	2%	2%	2%
d_1	1.23	0.15	-0.17	0.13
d_2	0.61	-0.52	-1.04	-1.28
$N(d_1)$	0.86590	0.56034	0.43147	0.55071
$N(d_2)$	0.61503	0.30189	0.14857	0.09956
$N'(d_1)$	0.18700	0.39437	0.39304	0.39572
$N'(d_2)$	0.33067	0.34868	0.23167	0.17493

(e) (*2 points*)

(i) Calculate the gain or loss of option E^*.

The response for this part is to be provided in the Excel spreadsheet.

(ii) Demonstrate how option E^* is an effective vehicle to take position on volatility convexity, given the result in part (e)(i).

The response for this part is required on the paper provided to you.

The responses for all parts of this question are required on the paper provided to you.

13.

(*5 points*) Assume a world with a risk-free bond, *B*, and an infinite number of stocks that are all simultaneously correlated with the entire market.

- Security M is tradable, and it tracks the behavior of the entire market.
- Stock S_i is one of the infinite number of stocks available in this market.
- The expected return and volatility are μ_M and σ_M for security M, respectively
- The expected return and volatility are μ_i and σ_i for stock S_i, respectively.
- The correlation between S_i and M is ρ_i.

(a) (*2 points*)

(i) Describe the following risk mitigating methods: dilution, diversification, and hedging.

(ii) Identify the situations where each method would be efficient.

(b) (*1 point*) Explain the effect of diluting risks on the Sharpe ratio.

(c) (*0.5 points*) Describe how to create a market-neutral portfolio S_i' using stock S_i and security M.

Assume there are infinite number of market-neutral portfolio S_i' which are uncorrelated to each other.

- The risk-free rate r is 2%;
- The beta β_i of stock S_i is 3;
- The expected return μ_i of stock S_i is 8%;
- Volatility σ_i of stock S_i is 120%;
- The volatility σ_M of the market is 30%.

(d) (*1.5 points*) Calculate the expected market return μ_M, and the Sharpe ratio of any market-neutral portfolio S_i'.

The responses for all parts of this question are required on the paper provided to you.

14.

(*6 points*) Your company sells single-premium Variable Annuity (VA) with a guaranteed minimum death benefits (GMDB) and a guaranteed lifetime withdrawal benefits (GLWB) rider options.

The following features are offered in the products:

- Death Benefit = Max (Premiums paid, Account Value)
- GLWB base grows at the maximum of 5% compound roll-up and annual ratchet for up to 10 years or until first withdrawal, whichever comes first, and annual ratchet thereafter.

(a) (*1 point*)

(i) Identify the option embedded in the VA with GLWB rider.

(ii) Explain when the option in part (a)(i) will be in-the-money (ITM).

In determining the cost of GLWB, you run 1,000 risk-neutral equity-only scenarios and the cost of GLWB is the average present values of the GLWB claims over all the scenarios projected. Based on the policy contract, the first withdrawal of the GLWB benefits needs to occur within 30 years. For each scenario, the account value is projected over a 30-year time horizon. When the account value falls to zero within the 30 years, the GLWB claim is calculated as the present value of annual guaranteed benefits over the policyholder's lifetime. When the account value is positive at the end of 30 years, GLWB claims become 0.

(b) (*1.5 points*) Critique the above approach with respect to the following aspects:

(i) The use of risk-neutral scenarios

(ii) The number of scenarios

(iii) The methodology of calculating the cost of GLWB

14. Continued

Now, your company would like to test the sensitivity of the cost of the GLWB under more adverse lapse dynamics. The following are the base and sensitivity dynamic lapse factors:

Base	Max [0, 100% – 1 * (GLWB ITM % - 100%)]
Sensitivity	Max [10%, 100% -0.75 * (GLWB ITM% - 110%)]

where GLWB ITM% = $\frac{PV\ of\ GLWB}{AV}$. For example,

GLWB ITM%	**Dynamic Lapse Factor**	
	Base	**Sensitivity**
0%	200%	183%
120%	80%	93%
140%	60%	78%
160%	40%	63%
180%	20%	48%
200%	0%	33%
220%	0%	18%
240%	0%	10%
260%	0%	10%

Below are the sensitivity test results:

	Average cost of GLWB (% of Benefit Base)
Base	47.1 bps
Sensitivity	55.8 bps

(c) (*2 points*) Assess and justify the reasonableness of the dynamic lapse factors of the base and sensitivity assumptions.

(d) (*1 point*) Explain and justify whether you agree with the sensitivity test results.

14. **Continued**

Your company is considering the addition of a new equity fund (albeit aggressive growth). Existing funds are one conservative equity fund, one moderate growth equity fund and a bond fund. The volatility of the bond fund is lower than the conservative equity fund. The table below summarizes the sensitivity test results of replacing the moderate growth equity fund with the new equity fund assuming other fund allocation assumptions remain the same:

Sensitivity test results of fund replacement
(Average cost of GLWB % of Benefit Base)

Age	Base	Sensitivity
Under 65	78.7 bps	92.7 bps
65 and above	48.7 bps	58.6 bps

(e) (*0.5 points*) Draw a conclusion from the above results.

The responses for all parts of this question are required on the paper provided to you.

15.

(*7 points*) Suppose that your company sells GMAB products and, under the Black-Scholes-Vasicek (BSV) model, the value of the liability guarantee is

$$\Omega_t = \psi\left(t, T, A_t, G_T\right)\, {}_{T-t}p_{x+t}$$

where

$\psi\left(t, T, A_t, G_T\right)$ = time-*t* price of a put option with maturity *T* and strike price G_T and written on account value $A_t \geq 0$

${}_{T-t}p_{x+t}$ = probability of a policyholder, who is *(x+t)* years old, surviving in the next $(T-t)$ years = ${}_{T-t}p^{(d)}_{x+t}\; {}_{T-t}p^{(w)}_{x+t}$

You are given:

$$\psi\left(t, T, A_t\, G_T\right) = G_T P_{t,T} \Phi\left(-d_1 + v\left(t, T\right)\right) - A_t e^{-a(T-t)} \Phi\left(-d_1\right)$$

$$v^2\left(t, T\right) = V\left(t, T\right) + \sigma_S^2 \left(T-t\right) + 2\varrho \frac{\sigma_S \sigma_r}{m}\left[\left(T-t\right) - \frac{1}{m}\left(1 - e^{-m(T-t)}\right)\right]$$

$$V\left(t, T\right) = \frac{\sigma_r^2}{m^2}\left[\left(T-t\right) + \frac{2}{m} e^{-m(T-t)} - \frac{1}{2m} e^{-2m(T-t)} - \frac{3}{2m}\right]$$

$$d_1 = \frac{\log\left[\dfrac{A_t}{G_T P_{t,T}}\right] - a\left(T-t\right) + \dfrac{v^2(t,T)}{2}}{v\left(t, T\right)}$$

where

$P_{t,T}$ = time-t price of a zero-coupon bond with a term-to-maturity of (T-t) year

σ_s = volatility of the Black-Scholes equity model

σ_r = volatility of the Vasicek interest rate model

ϱ = correlation between Brownian motions of Black-Scholes and Vasicek models

a = rider fee for the GMAB guarantee

m = mean reversion parameter of the Vasicek interest rate model

mortality rate = 1% per year

lapse rate = base lapse * dynamic lapse multiple, where base lapse = 2%

15. Continued

$$Dynamic\ Lapse\ Multiple = f\left(ITM\right)$$

$$= \begin{cases} 0.1 & , \quad 2.3 < ITM \\ 1 - 0.75 \cdot \left(ITM - 1.1\right) & , \quad 1.1 < ITM \leq 2.3 \\ 1 & , \quad ITM \leq 1.1 \end{cases}$$

where $ITM = \dfrac{G_T}{A_t}$

(a) (*1 point*) Show that the value of the liability guarantee is

$$\Omega_t = \psi\left(t, T, A_t, G_T\right)\left[0.99 \cdot \left(0.9635 + 0.015 \cdot ITM\right)\right]^{T-t}, \qquad if\ 1.1 < ITM < 2.3$$

You are given $\dfrac{\partial \psi\left(t, T, A_t, G_T\right)}{\partial A_t} = -e^{-a(T-t)}\Phi\left(-d_1\right)$.

(b) (*3 points*) Show that, for $1.1 < ITM < 2.3$, the Delta of the value of the liability guarantee is:

$$\frac{\partial \Omega_t}{\partial A_t} = -e^{-a(T-t)}\Phi\left(-d_1\right)\frac{\Omega_t}{\psi\left(t, T, A_t, G_T\right)} - \left(T - t\right)\left(\frac{\Omega_t\left(0.015 \cdot ITM\right)}{A_t\left[\left(0.9635 + 0.015 \cdot ITM\right)\right]}\right)$$

(c) (*3 points*) Determine, for $1.1 < ITM < 2.3$ using results from part (b), whether the absolute magnitude of the Delta of the value of the liability guarantee is larger with or without dynamic lapse multiple.

****END OF EXAMINATION****